Advanced topics in string and field theory: Complex manifolds and Calabi-Yau manifolds

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7 Cohomology on complex and Kähler manifolds

In this chapter we will study cohomology on complex and especially Kähler manifolds, where we will refine deRham cohomology using the Dolbeault operators we introduced when first encountering complex manifolds.

7.1 Dolbeault cohomology

We saw that on complex manifolds we can define nilpotent operators ∂ and $\bar{\partial}$. We can use these to define two kinds of so-called Dolbeault cohomologies: one for ∂ and one for $\bar{\partial}$. Convention has it that the cohomology of $\bar{\partial}$ is usually called the "Dolbeault" cohomology. Let us first study this cohomology for complex manifolds before specialising to Kähler manifolds. As the construction is largely analogous to that for deRham cohomology most of the details are left as exercises.

We begin by defining an inner product for p,q-forms.

Definition: Let (M,J,g) be a complex manifold with *Hermitian* metric g. Then we define $(\,,)_{p,q}$ to be an inner product of p,q-forms, i.e.

$$(\,,)_{p,q} : \Omega^{p,q}(M) \otimes \Omega^{p,q}(M) \longrightarrow \mathbb{C}\,, \tag{7.1}$$

such that for $\alpha\,,\beta \in \Omega^{p,q}(M)$ we have

$$(\alpha,\beta)_{p,q} = \int_M \alpha \wedge \star\bar{\beta}\,. \tag{7.2}$$

Here $\bar{\beta}$ is the complex conjugate of β.

Theorem 7.1: The inner product $(\,,\,)_{p,q}$ satisfies

$$(\alpha, \alpha)_{p,q} \geq 0\,, \tag{7.3}$$

for all $\alpha \in \Omega^{p,q}(M)$, with equality iff $\alpha = 0$.

Proof: This follows immediately from the result of the exercise below.

Exercise 7.1: Use the fact that on a $2n$-dimensional complex manifold we can write

$$\epsilon^{a_1 \dots a_n \bar{b}_1 \dots \bar{b}_n} = \epsilon^{a_1 \dots a_n} \epsilon^{\bar{b}_1 \dots \bar{b}_n}\,, \tag{7.4}$$

where $\epsilon^{a_1 \dots a_n} = \epsilon^{\bar{a}_1 \dots \bar{a}_n} = \pm |g|^{1/4}$, with the sign depending whether $(a_1, \dots, a_n)$ is an even or odd permutation of $(1, \dots, n)$, to show that

$$(\alpha, \beta)_{p,q} = \int_M \alpha_{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q} \bar{\beta}^{a_1 \dots a_p \bar{b}_1 \dots \bar{b}_q} \sqrt{g}\; d^n z d^n \bar{z}\,. \tag{7.5}$$

We can now define the adjoint $\bar{\partial}^\dagger$ of the operator $\bar{\partial}$.

Definition: The adjoint $\bar{\partial}^\dagger : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q-1}(M)$ is defined as

$$\left(\alpha, \bar{\partial}\beta\right)_{p,q} = \left(\bar{\partial}^\dagger \alpha, \beta\right)_{p,q}\,, \tag{7.6}$$

for all $\alpha \in \Omega^{p,q}(M)$ and $\beta \in \Omega^{p,q-1}(M)$.

Theorem 7.2:

$$\bar{\partial}^\dagger = - \star \bar{\partial} \star\,. \tag{7.7}$$

Exercise 7.2: Prove the above.
Hint: You may wish to first show that for any p,q-form $\alpha \in \Omega^{p,q}(M)$

$$\bar{\partial}(\alpha \wedge \beta) = \bar{\partial}\alpha \wedge \beta + (-1)^{p+q} \alpha \wedge \bar{\partial}\beta\,. \tag{7.8}$$

We can now introduce a Hodge operator as before and discuss harmonic forms with respect to $\bar{\partial}$.

Definition: Define

$$\Delta_{\bar{\partial}} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q}(M) \,, \tag{7.9}$$

by

$$\Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial} \,. \tag{7.10}$$

Let us introduce some more notation.

Definition:

- $Z^{p,q}_{\bar{\partial}}$ ($\bar{Z}^{p,q}_{\bar{\partial}}$) is the space of (co-)closed p,q-forms with respect to $\bar{\partial}$.
- $B^{p,q}_{\bar{\partial}}$ ($\bar{B}^{p,q}_{\bar{\partial}}$) is the space of (co-)exact p,q-forms with respect to $\bar{\partial}$.
- $H^{p,q}_{\bar{\partial}} = Z^{p,q}_{\bar{\partial}}/B^{p,q}_{\bar{\partial}}$ is the (p,q)-th cohomology group with respect to $\bar{\partial}$.
- $\mathcal{H}^{p,q}_{\bar{\partial}}$ is the space of harmonic p,q-forms with respect to $\Delta_{\bar{\partial}}$.

Theorem 7.3: We can perform a Hodge decomposition with respect to the Dolbeault operators, i.e. we can write any p,q-form ω as

$$\omega = \alpha + \bar{\partial}\beta + \bar{\partial}^\dagger\gamma \,, \tag{7.11}$$

where $\alpha \in \mathcal{H}^{p,q}_{\bar{\partial}}$, i.e. $\Delta_{\bar{\partial}}\alpha = 0$. Thus, we can write as before

$$H^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q}_{\bar{\partial}} \oplus B^{p,q}_{\bar{\partial}} \oplus \bar{B}^{p,q}_{\bar{\partial}} \,. \tag{7.12}$$

Corollary: $\mathcal{H}^{p,q}_{\bar{\partial}}$ is isomorphic to $H^{p,q}_{\bar{\partial}}$.

For a complex manifold the Dolbeault cohomology classes and deRham cohomology classes are different. In particular, the former depend on the choice of complex structure (as this determines what we call (p,q)-forms) while deRham cohomology is topological.

However, for a Kähler manifold the two cohomologies are equivalent. This can be seen by explicitly computing the three Laplacians $\Delta_{\bar{\partial}}$, Δ_{∂} and Δ. One finds

Theorem 7.4: For a Kähler manifold,

$$\Delta = 2\Delta_{\bar{\partial}} = 2\Delta_{\partial} \,. \tag{7.13}$$

Proof: See exercises below.

Exercise 7.3: Show that for a Kähler manifold $\bar{\partial}^\dagger$ can be written in terms of the covariant derivative

$$d^\dagger\omega = \frac{(-1)^{1+p+q}}{p!\,(q-1)\,!}\nabla^c\omega_{a_1\ldots a_p c\bar{b}_1\ldots\bar{b}_{q-1}}dz^{a_1}\wedge\ldots\wedge dz^{a_p}\wedge d\bar{z}^{b_1}\wedge\ldots\wedge d\bar{z}^{b_{q-1}}\,. \tag{7.14}$$

Exercise 7.4: Prove (7.13) using the result of the previous exercise.

Theorem 7.5: Let (M, J, g) be a Kähler manifold. Then

$$\mathcal{H}^k = \bigoplus_{j=0}^{k}\mathcal{H}_{\bar{\partial}}^{j,k-j}\,. \tag{7.15}$$

Definition: We define the **Hodge numbers**,

$$h^{p,q} = \dim H_{\bar{\partial}}^{p,q}\,, \tag{7.16}$$

to be the dimension of the (p,q)-th cohomology group.

Using the decomposition of the deRham cohomology groups in terms of Dolbeault cohomology groups we find:

Corollary: Let (M, J, g) be a Kähler manifold. Then the Betti numbers and Hodge numbers satisfy

$$b^k = \sum_{j=0} h^{j,k-j}\,. \tag{7.17}$$

Theorem 7.6: On a $2n$-dimensional complex manifold, the Hodge numbers satisfy the following identities

$$\begin{aligned} h^{j,k} &= h^{k,j}\,,\\ h^{j,k} &= h^{n-j,n-k}\,. \end{aligned} \tag{7.18}$$

Proof: The first identity arises because the complex conjugate of a harmonic (p,q)-form is a harmonic (q,p)-form. The second comes from applying Hodge duality.

It is customary to arrange the Hodge numbers in a so-called **Hodge diamond**. For a $2n$-

dimensional complex manifold, the Hodge diamond is given by

$$\begin{array}{ccccccccc} & & & & h^{0,0} & & & & \\ & & & h^{1,0} & & h^{0,1} & & & \\ & & h^{2,0} & & h^{1,1} & & h^{0,2} & & \\ & \iddots & & & \vdots & & & \ddots & \\ h^{n,0} & h^{n-1,1} & & & \cdots & & h^{1,n-1} & & h^{0,n} \\ & \ddots & & & \vdots & & & \iddots & \\ & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\ & & & h^{n,n-1} & & h^{n-1,n} & & & \\ & & & & h^{n,n} & & & & \end{array} . \tag{7.19}$$

Because of the identities discussed not all elements are independent. For example, for a 1-dimensional complex manifold the Hodge diamond is given by

$$\begin{array}{ccc} & h^{0,0} & \\ h^{1,0} & & h^{1,0} \\ & h^{0,0} & \end{array} , \tag{7.20}$$

and we see there are only 2 independent hodge numbers. Similarly for a 2-dimensional complex manifold we would have

$$\begin{array}{ccccc} & & h^{0,0} & & \\ & h^{1,0} & & h^{1,0} & \\ h^{2,0} & & h^{1,1} & & h^{2,0} \\ & h^{1,0} & & h^{1,0} & \\ & & h^{0,0} & & \end{array} , \tag{7.21}$$

and there are only 4 independent Hodge numbers.

7.2 Hodge numbers of Kähler manifolds

Let us now consider Kähler manifolds in particular. We can find some topological obstructions to Kähler manifolds by the following considerations.

Theorem 7.7: The Kähler form is harmonic.

Proof: Recall that on a Kähler manifold there exists a globally well-defined closed $(1,1)$-form ω, the Kähler form, i.e.

$$d\omega = 0\,. \tag{7.22}$$

We also saw that the connection is compatible with the Kähler form, i.e.

$$\nabla\omega = 0\,. \tag{7.23}$$

From the definition of $d^\dagger$ in the previous chapter we see that ω is co-closed. Thus ω is harmonic and this completes our proof.

Corollary: $h^{1,1} > 0$ and hence $b^2 > 0$.

Definition: On a Kähler manifold, the **Kähler class** $[\omega]$ is the cohomology class of the Kähler form ω.

Theorem 7.8: On a compact, closed Kähler manifold $h^{k,k} > 0$, $\forall\ 0 \leq k \leq n$, and hence $b^{2k} > 0$, $\forall\ 0 \leq k \leq n$.

Proof: Using the Kähler form we can form a closed (k,k)-form by wedging the Kähler form with itself k times:

$$\omega^k = \omega \wedge \ldots \wedge \omega\,. \tag{7.24}$$

Since $d\omega = 0$ we have $\bar\partial\omega = 0$ and hence

$$\bar\partial\omega^k = 0\,. \tag{7.25}$$

Thus, $\omega^k \in H^{k,k}_{\bar\partial}$. Let us assume that it vanishes in the Dalboult cohomology class, i.e. that

$$\omega^k = \bar\partial\alpha\,, \tag{7.26}$$

for some $\alpha \in \Omega^{k,k-1}$. Now note that on a $2n$-dimensional Kähler manifold $\omega^n = n!\,\epsilon$ where ϵ is the volume form. You can see this easily in normal complex coordinates. Then we can write

$$\epsilon = \frac{1}{n!}\omega^n = \frac{1}{n!}\omega^k \wedge \omega^{n-k} = \frac{1}{n!}\bar\partial\alpha \wedge \omega^{n-k}\,. \tag{7.27}$$

However, $\alpha \wedge \omega^{n-k}$ is a $(n, n-1)$-form and so $\partial\alpha \wedge \omega^{n-k} = 0$. Thus,

$$\epsilon = \frac{1}{n!}d\alpha \wedge \omega^{n-k}\,, \tag{7.28}$$

But if we integrate the volume form over the manifold we get the volume

$$V = \int \epsilon = \int d\alpha \wedge \omega^{n-k} = 0 \tag{7.29}$$

by Stoke's Theorem. But this is not possible for a compact closed manifold and hence ω^k must not be in a vanishing Dalboult cohomology class. This completes the proof.

Theorem 7.9: On a Kähler manifold, all odd Betti numbers b^{2k-1} (for all $0 \leq 2k-1 \leq n$) are even.

Proof:

$$
\begin{aligned}
b^{2k-1} &= \sum_{j=0}^{2k-1} h^{j,2k-j-1} = \sum_{j=0}^{k-1} h^{j,2k-j-1} + \sum_{j=k}^{2k-1} h^{j,2k-j-1} \\
&= \sum_{j=0}^{k-1} h^{j,2k-j-1} + \sum_{j=0}^{k-1} h^{k+1,k-j-1} \\
&= \sum_{j=0}^{k-1} \left(h^{j,2k-j-1} + h^{2k-j-1,j}\right) = 2\sum_{j=0}^{k-1} h^{j,2k-j-1}\,,
\end{aligned}
\tag{7.30}
$$

where in the penultimate equality we used the relabelling $j \to k-1-j$.

Topological requirements such as these are an easy way to quickly determine whether a manifold is Kähler or not.

Example 7.1: Products of odd spheres, $M_{r,s} = S^{2r+1} \times S^{2s+1}$, are not Kähler.

Proof: We will show that there are no harmonic 2-forms on $M_{r,s}$. If a harmonic 2-form ω existed on $M^{r,s}$ it would have to be of the form

$$
\omega = \alpha_2 + \alpha_1 \wedge \beta_1 + \beta_2\,, \tag{7.31}
$$

where $\alpha_2 \in \mathcal{H}^2(S^{2r+1})$, $\beta_2 \in \mathcal{H}^2(S^{2s+1})$, $\alpha_1 \in \mathcal{H}^1(S^{2r+1})$ and $\beta_1 \in \mathcal{H}^1(S^{2s+1})$ are harmonic 2-forms and 1-forms on the respective spheres. However, we have seen in chapter 6 that n-spheres S^n have $b_0 = b_n = 1$ and all other Betti numbers vanish. Thus $\omega = 0$. This completes the proof.

Before moving on to Calabi-Yau manifolds, let us quickly re-examine the first Chern class. Recall that

$$
c_1 = [\frac{1}{2\pi}\mathcal{R}]\,, \tag{7.32}
$$

where $\mathcal{R} = -i\partial\bar{\partial}\ln\sqrt{g}$ is the Ricci-form.

Theorem 7.10: Under a smooth variation of the metric $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$ the first Chern class is invariant.

Proof: Under the variation, the Ricci form becomes

$$
\mathcal{R}' = \mathcal{R} - \frac{i}{2}\partial\bar{\partial}\left(g^{mn}\delta g_{mn}\right)\,. \tag{7.33}
$$

Now recall that $\partial\bar{\partial} = \frac{1}{2}d\left(\partial - \bar{\partial}\right)$ and thus we see that

$$\mathcal{R}' = \mathcal{R} - d\mathcal{A}\,, \tag{7.34}$$

where $\mathcal{A} = \frac{i}{4}\left(\partial - \bar{\partial}\right) g^{mn}\delta g_{mn}$ is a well-defined 1-form since $g^{mn}\delta g_{mn}$ is a coordinate scalar.

Exercise 7.5: Show that under a smooth variation of the metric, the Ricci form transforms as in equation (7.33).

We are now in a strong position to study Calabi-Yau manifolds which will be the topic of our final lecture.