Quantum Field Theory
Example Sheet 2
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Exercise 1

We will use the convetion that the Lorentz generators are

$$S^{\mu\nu} = \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right] . \tag{1}$$

Note this implies that the finite transformations would be

$$D\left[\Lambda\right]^\alpha{}_\beta = \exp\left[\frac{1}{2}\Omega_{\mu\nu}\left(S^{\mu\nu}\right)^\alpha{}_\beta\right] \tag{2}$$

The object is to show that the $S^{\mu\nu}$ defined below generate a representation of the Lie algebra $\mathfrak{so}(3,1)$, the Lie algebra of the Lorentz group $\mathrm{SO}(3,1)$.

$$\begin{aligned}
\left[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu\right] &= \gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\
&= \gamma^\kappa\left\{\gamma^\lambda, \gamma^\mu\right\}\gamma^\nu - \gamma^\kappa\gamma^\mu\gamma^\lambda\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\
&= 2g^{\lambda\mu}\gamma^\kappa\gamma^\nu - \left\{\gamma^\kappa, \gamma^\mu\right\}\gamma^\lambda\gamma^\nu + \gamma^\mu\gamma^\kappa\gamma^\lambda\gamma^\nu - \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\
&= 2g^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2g^{\kappa\mu}\gamma^\lambda\gamma^\nu + \gamma^\mu\gamma^\kappa\left\{\gamma^\nu, \gamma^\lambda\right\} - \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\
&= 2g^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2g^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2g^{\mu\kappa}\gamma^\nu\gamma^\lambda - \gamma^\mu\left\{\gamma^\kappa, \gamma^\nu\right\}\gamma^\lambda + \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda - \gamma^\mu\gamma^\nu\gamma^\kappa\gamma^\lambda \\
&= 2g^{\lambda\mu}\gamma^\kappa\gamma^\nu - 2g^{\kappa\mu}\gamma^\lambda\gamma^\nu + 2g^{\mu\kappa}\gamma^\nu\gamma^\lambda - 2g^{\kappa\nu}\gamma^\mu\gamma^\lambda
\end{aligned}$$

Now, to perform the main computation of interest, write $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$ in a slightly more convenient form,

$$\begin{aligned}
S^{\kappa\lambda} = \frac{1}{4}[\gamma^\kappa, \gamma^\lambda] &= \frac{1}{4}\left\{\gamma^\kappa\gamma^\lambda - \gamma^\lambda\gamma^{ka}\right) \\
&= \frac{1}{4}\left(\gamma^\kappa\gamma^\lambda - \left\{\gamma^\lambda, \gamma^\kappa\right\} + \gamma^\kappa\gamma^\lambda\right) \\
&= \frac{1}{2}\left(\gamma^\kappa\gamma^\lambda - g^{\kappa\lambda}\right) .
\end{aligned}$$

This allows us to calculate that

$$\begin{aligned}
\left[S^{\kappa\lambda}, S^{\mu\nu}\right] &= \frac{1}{2}[\gamma^\kappa\gamma^\lambda, \gamma^\mu\gamma^\nu] \\
&= \frac{1}{2}\left(g^{\lambda\mu}\gamma^\kappa\gamma^\nu - g^{\kappa\mu}\gamma^\lambda\gamma^\nu + g^{\mu\kappa}\gamma^\nu\gamma^\lambda - g^{\kappa\nu}\gamma^\mu\gamma^\lambda\right) \\
&= g^{\lambda\mu}S^{\kappa\nu} + \frac{1}{2}g^{\lambda\mu}g^{\kappa\nu} - g^{\mu\kappa}S^{\lambda\nu} - \frac{1}{2}g^{\mu\kappa}g^{\lambda\nu} + g^{\lambda\nu}S^{\mu\kappa} + \frac{1}{2}g^{\lambda\nu}g^{\mu\kappa} - g^{\kappa\nu}S^{\mu\lambda} - \frac{1}{2}g^{\kappa\nu}g^{\mu\lambda} \\
&= g^{\lambda\mu}S^{\kappa\nu} - g^{\mu\kappa}S^{\lambda\nu} + g^{\lambda\nu}S^{\mu\kappa} - g^{\kappa\nu}S^{\mu\lambda}
\end{aligned} \tag{3}$$

The $S^{\mu\nu}$ thus define a representation of $\mathfrak{so}(3,1)$, called the spin representation - it is not irreducible (see next exercise).

Exercse 2

We use the brackets (3) worked out in the previous exercise, but first write

$$\begin{aligned}
S^i &= \frac{i}{4}\epsilon^i{}_{jk}\gamma^j\gamma^k = \frac{i}{4}\epsilon^i{}_{jk}\left(\gamma^j\gamma^k - \delta^{jk}\right) \\
&= \frac{i}{2}\epsilon^i{}_{jk}S^{jk},
\end{aligned}$$

where skew symmetry of ϵ_{ijk} allows us to insert the symmetric term δ^{ij}. Thus

$$\begin{aligned}
[S_i, S_j] &= -\frac{1}{4}\epsilon_{ikl}\epsilon_{jmn}[S^{kl}, S^{mn}] \\
&= -\frac{1}{4}\epsilon_{ikl}\epsilon_{jmn}\left(\delta^{lm}S^{kn} - \delta^{km}S^{ln} + \delta^{ln}S^{mk} - \delta^{kn}S^{ml}\right) \\
&= -\frac{1}{4}\left(\epsilon_{ikl}\epsilon_{jln}S^{kn} - \epsilon_{ikl}\epsilon_{ikn}S^{ln} + \epsilon_{ikl}\epsilon_{jml}S^{mk} - \epsilon_{ikl}\epsilon_{jmk}S^{ml}\right).
\end{aligned}$$

Relabeling indices in the final three terms and permuting in the alternating symbols, we have

$$\begin{aligned}
[S_i, S_j] &= -\frac{1}{4}\left(\epsilon_{ikl}\epsilon_{jln}S^{kn} - \epsilon_{ilk}\epsilon_{iln}S^{kn} + \epsilon_{ikl}\epsilon_{jnl}S^{nk} - \epsilon_{iln}\epsilon_{jnl}S^{nk}\right) \\
&= -\epsilon_{ikl}\epsilon_{jln}S^{kn} \\
&= \epsilon_{ikl}\epsilon_{jnl}S^{kn} \\
&= \left(\delta_{ij}\delta_{kn} - \delta_{in}\delta_{kj}\right)S^{kn} \\
&= -S_{ji} \\
&= S_{ij},
\end{aligned}$$

where in the final two steps the obvious skew-symmetry $S^{kn} = -S^{nk}$ was employed, so that, in the penultimate step, $S^{kn}\delta_{kn} = 0$. To obtain the desired result, consider

$$\begin{aligned}
i\epsilon_{ijk}S^k &= -\frac{1}{2}\epsilon_{ijk}\epsilon_{kln}S^{ln} = \frac{1}{2}\epsilon_{ijk}\epsilon_{nlk}S^{ln} = \frac{1}{2}\left(\delta_{im}\delta_{jl} - \delta_{il}\delta_{jm}\right)S^{ln} \\
&= \frac{1}{2}\left(S_{ij} - S_{ji}\right) \\
&= S_{ij}
\end{aligned}$$

Thus

$$[S_i, S_j] = i\epsilon_{ijk}S^k,$$

and we see that the S^i furnish a representation of the Lie algebra of the rotation group in three dimensions, $\mathfrak{so}(3) \cong \mathfrak{su}(2)$.

Since $\gamma^0\gamma^i = -\gamma^i\gamma^0$,

$$\gamma^0 S^i = \frac{i}{4}\epsilon_{ijk}\gamma^0\gamma^j\gamma^k = +\frac{i}{4}\epsilon_{ijk}\gamma^j\gamma^k\gamma^0 = S^i\gamma^0$$

i.e. $[\gamma^0, S^i] = 0$.

Using the claim below, the same reasoning demonstrates that $[\gamma^5, S^i] = 0$.

Claim:

$$\left\{\gamma^5, \gamma^\mu\right\} = 0 \tag{4}$$

This is straightforward to prove: since γ^μ anti-commutes with γ^ν whenever $\mu \neq \nu$, we can commute γ^μ past the three γ^ν in γ^5 for which $\mu \neq \nu$ and pick up a factor of $(-1)^3 = -1$.

Mathematical Remark: In fact, the previous argument demonstrates that $[\gamma^5, S^{\mu\nu}] = 0$, which in turn tells us that the representation of $\mathfrak{so}(3,1)$ furnished by the $S^{\mu\nu}$ is reducible: since $(\gamma^5)^2 = 1$ (see Exercise 10) it follows that the space of Dirac spinors (i.e. $\mathbb{C}^4$) splits into eigensubspaces of γ^5 with eigenvalues ± 1; the identity $[\gamma^5, S^{\mu\nu}] = 0$ shows that the action of the $S^{\mu\nu}$ preserves this eigenspace decomposition, i.e. sends all Dirac spinors of γ^5 eigenvalue ± 1 into a spinor with the same eigenvalue. These eigenspaces are thus proper invariant subspaces, and so the representation is reducible.
More generally, this is a feature of **even** dimensions - in **odd** dimensions the representation defined here is indeed irreducible. See Michelson & Lawson, *Spin Geometry* for full details.

Fixing $i \in \{1, 2, 3\}$, there are $j, k \in \{1, 2, 3\}$ uniquely defined by $j, k \neq i$ and $j < k$. Then, without any implicit summations,

$$S_i = \frac{i}{4}\epsilon_{ijk}\left(\gamma^j\gamma^k - \gamma^k\gamma^j\right).$$

Then as $\epsilon_{ijk} = \pm 1$, $(\epsilon_{ijk})^2 = +1$, so

$$(S^i)^2 = -\frac{1}{16}\left(\gamma^j\gamma^k\gamma^j\gamma^k + \gamma^k\gamma^j\gamma^k\gamma^j - \gamma^j(\gamma^k)^2\gamma^j - \gamma^k(\gamma^j)^2\gamma^k\right).$$

Now, in the first two terms use that $\gamma^j\gamma^k = -\gamma^k\gamma^j$, whenever $j \neq k$, so that, as $(\gamma^l)^2 = -\mathrm{I}$,

$$S^i = -\frac{1}{16}\left(-(\gamma^j)^2(\gamma^k)^2 - (\gamma^k)^2(\gamma^j)^2 - \gamma^j(\gamma^k)^2\gamma^j - \gamma^k(\gamma^j)^2\gamma^k\right) = -\frac{1}{16}\left(-4\,\mathrm{I}\right) = \frac{1}{4}\,\mathrm{I}$$

Consider the following 4-dimensional representation of the Clifford algebra

$$\gamma^0 = \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where, for reference,

$$(\sigma^i) = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

and these satisfy

$$\sigma^i\sigma^j = \delta^{ij}\mathrm{I}_2 + i\epsilon^{ijk}\sigma_k, \tag{5}$$

so that, in particular,

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}\mathrm{I}_2\ ,\ [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma_k. \tag{6}$$

The S^i having the following matrix representatives

$$S_i = \frac{i}{4}\epsilon_{ijk}\gamma^j\gamma^k = \frac{i}{4}\epsilon_{ijk}\begin{pmatrix} -\sigma^j\sigma^k & 0 \\ 0 & -\sigma^j\sigma^k \end{pmatrix}.$$

According to (5),

$$\epsilon_{ijk}\sigma^j\sigma^k = i\epsilon_{ijk}\epsilon_{jkl}\sigma^l = i\left(\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl}\right)\sigma^l = 2i\sigma^i.$$

So that

$$S^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}.$$

Then by (6),

$$[S_i, S_j] = \frac{1}{4}\begin{pmatrix} [\sigma_i, \sigma_j] & 0 \\ 0 & [\sigma_i, \sigma_j] \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2i\epsilon_{ijk}\sigma^k & 0 \\ 0 & 2i\epsilon_{ijk}\sigma^k \end{pmatrix} = i\epsilon_{ijk}\left\{\frac{1}{2}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}\right\} = i\epsilon_{ijk}S^k,$$

as desired.

From (6), $(\sigma^i)^2 = +1$, so

$$(S^i)^2 = \frac{1}{4}\begin{pmatrix} (\sigma^i)^2 & 0 \\ 0 & (\sigma^i)^2 \end{pmatrix} = \frac{1}{4}\,\mathrm{I}_2$$

$$\begin{aligned}\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 &= i\begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix}\begin{pmatrix} 0 & \sigma^1 \\ -\sigma^1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \\ &= i\begin{pmatrix} 0 & -\sigma^1\sigma^2\sigma^3 \\ -\sigma^1\sigma^2\sigma^3 & 0 \end{pmatrix}\end{aligned}$$

$$\sigma^1\sigma^2\sigma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\,\mathrm{I}_2$$

$$\gamma^5 = \begin{pmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix}$$

$$[\gamma^5, S^i] = \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} - \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}\right] = 0$$

Likewise, $[\gamma^0, S^i] = 0$.

Since $\mathfrak{so}(3)$ (the Lie algebra of the rotation group, SO(3)) is a sub-algebra of the Lorentz algebra, these calculations tell us that by restricting the above representation of $\mathfrak{so}(1,3)$ to a representation of $\mathfrak{so}(3)$, we obtain two copies of the usual spin half representation encountered in non-relativistic physics. In other words, Dirac spinors furnish the (reducible) representation of $\frac{1}{2} \oplus \frac{1}{2}$ of $\mathfrak{so}(3)$ and thus have spin $\frac{1}{2}$.

Notice that although we have exhibited the representation of $\mathfrak{so}(3) \leq \mathfrak{so}(1,3)$ as a sum of irreducibles, i.e. the matrices S^i generating $\mathfrak{so}(3)$ are block diagonal, this does not show is explicitly that the representation of $\mathfrak{so}(1,3)$ constructed here is reducible (as claimed above) since the S^{0i} are in fact not block diagonal in this representation (**additional exercise:** show this).

A representation where γ^5 is block diagonal ensures all $S^{\mu\nu}$ are block diagonal as well. The so called *chiral representation* with

$$\gamma^0 = \begin{pmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \tag{7}$$

achieves this (**additional exercise:** show this).

Exercise 3

As the identity $(\gamma^5)2 = \mathrm{I}$ is used in numerous places throughout the following calculations, it is more logical to prove this first. Picking up various factors of -1 from commuting factors, we have

$$\begin{aligned}
(\gamma^5)^2 &= -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 \\
&= -(-1)^3(\gamma^0)^2\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 \\
&= +(-1)^2(\gamma^0)^2(\gamma^1)^2\gamma^2\gamma^3\gamma^2\gamma^3 \\
&= -(\gamma^0)^2(\gamma^1)^2(\gamma^2)^2(\gamma^3)^2,
\end{aligned}$$

and, finally, using $(\gamma^0)^2 = +\mathrm{I}$, $(\gamma^i)^2 = -\mathrm{I}$, we see that $(\gamma^5)^2 = \mathrm{I}$.

1. We prove instead that

   $$\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right) = 0\ ,\ k \in \mathbb{N},$$

   as this accounts for this Exercise and Exercises 3 and 8 also.

   We begin by inserting a factor of $(\gamma^5)2 = \mathrm{I}$ into the far right of the trace and use the cyclicity of the trace to move one of these factors of γ^5 to the far left,

   $$\begin{aligned}
   \mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right) &= \mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}(\gamma^5)2\right) \\
   &= \mathrm{Tr}\left(\gamma^5\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\gamma^5\right).
   \end{aligned}$$

   Now use the identity (4) to begin anti-commuting this γ^5 past each factor of γ^{μ_i} to its right.

   $$\begin{aligned}
   \mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right) &= (-1)\mathrm{Tr}\left(\gamma^{\mu_1}\gamma^5\gamma^2\cdots\gamma^{\mu_{2k+1}}\gamma^5\right) \\
   &= (-1)^{2k+1}\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\gamma^5\gamma^5\right) \\
   &= (-1)^{2k+1}\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}(\gamma^5)2\right) \\
   &= -\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right) \\
   \Rightarrow \mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right) &= 0
   \end{aligned} \tag{8}$$

   where in the penultimate step, we use $(\gamma^5)^2 = \mathrm{I}$ again.

2. Splitting the product $\gamma^\mu\gamma^\nu$ into symmetric and skew-symmetric parts, we have

   $$\begin{aligned}
   \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) &= \mathrm{Tr}\left(\frac{1}{2}\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}[\gamma^\mu,\gamma^\nu]\right) \\
   &= \frac{1}{2}\mathrm{Tr}\,\{\gamma^\mu,\gamma^\nu\} + \frac{1}{2}\mathrm{Tr}\,[\gamma^\mu,\gamma^\nu]
   \end{aligned}$$

   The second term vanishes since the identity $\mathrm{Tr}(XY) = \mathrm{Tr}(YX)$ implies

   $$0 = \mathrm{Tr}(XY) - \mathrm{Tr}(YX) = \mathrm{Tr}\left(XY - YX\right) = \mathrm{Tr}\,[X,Y].$$

   Using the defining relation of the Clifford algebra, this then gives

   $$\begin{aligned}
   \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) &= \mathrm{Tr}\left(g^{\mu\nu}\mathrm{I}_4\right) \\
   &= 4g^{\mu\nu}
   \end{aligned} \tag{9}$$

5. Here, insert a factor of $(\gamma^0)^2 = \mathrm{I}$ and anti-commute as before

   $$\begin{aligned}
   \mathrm{Tr}\,\gamma^5 &= \mathrm{Tr}\left(\gamma^5(\gamma^0)^2\right) \\
   &= -\mathrm{Tr}\left(\gamma^0\gamma^5\gamma^0\right) \\
   &= -\mathrm{Tr}\left(\gamma^5(\gamma^0)^2\right) \\
   \Rightarrow \mathrm{Tr}\,\gamma^5 &= 0
   \end{aligned} \tag{10}$$

6. As was done previously, split the product $\gamma^\mu\gamma^\nu$ into symmetric and skew-symmetric parts and then use the Clifford relations,

$$\begin{aligned}
\not{p}\not{q} &= \frac{1}{2}p_\mu q_\nu \left(\{\gamma^\mu,\gamma^\nu\} + [\gamma^\mu,\gamma^\nu]\right) \\
&= \frac{1}{2}p_\mu q_\nu \left(2g^{\mu\nu}\mathrm{I} + 4S^{\mu\nu}\right) \\
&= p\cdot q\,\mathrm{I} + 2S^{\mu\nu}p_\mu q_\nu .
\end{aligned}$$

This time simply anti-commute factors,

$$\begin{aligned}
\not{p}\not{q} &= p_\mu q_\nu \gamma^\mu\gamma^\nu \\
&= p_\mu q_\nu \left(\{\gamma^\mu,\gamma^\nu\} - \gamma^\nu\gamma^\mu\right) \\
&= p_\mu q_\nu \left(2g^{\mu\nu}\,\mathrm{I} - \gamma^\nu\gamma^\mu\right) \\
&= 2p\cdot q - \not{q}\not{p}
\end{aligned}$$

7. Clearly, by (9),

$$\mathrm{Tr}(\not{p}\not{q}) = p_\mu q_\nu \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) = 4p_\mu q_\nu g^{\mu\nu} = 4p\cdot q$$

8. Clearly

$$\mathrm{Tr}\left(\not{p}1\cdots\not{p}^{2k+1}\right) = p1_{\mu_1}\cdots p^{2k+1}_{\mu_{2k+1}}\mathrm{Tr}\left(\gamma^{\mu_1}\cdots\gamma^{\mu_{2k+1}}\right),$$

which vanishes by (8).

9. As before, we use cyclicity of the trace and then perform a sequence of anti-commutations to move a factor back to its original position,

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) = \mathrm{Tr}\left(\gamma^\sigma\gamma^\mu\gamma^\nu\gamma^\rho\right) &= \mathrm{Tr}\left(\{\gamma^\sigma,\gamma^\mu\}\gamma^\nu\gamma^\rho - \gamma^\mu\gamma^\sigma\gamma^\nu\gamma^\rho\right) \\
&= 2g^{\sigma\mu}\mathrm{Tr}\left(\gamma^\nu\gamma^\rho\right) - \mathrm{Tr}\left(\gamma^\mu\{\gamma^\sigma,\gamma^\nu\}\gamma^\rho\right) + \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho\right) \\
&= 2g^{\sigma\mu}\mathrm{Tr}\left(\gamma^\nu\gamma^\rho\right) - 2g^{\sigma\nu}\mathrm{Tr}\left(\gamma^\mu\gamma^\rho\right) + \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\{\gamma^\sigma,\gamma^\rho\}\right) - \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) \\
\Rightarrow \mathrm{Tr}\left(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right) &= g^{\sigma\mu}\mathrm{Tr}\left(\gamma^\nu\gamma^\rho\right) - g^{\sigma\nu}\mathrm{Tr}\left(\gamma^\mu\gamma^\rho\right) + g^{\sigma\rho}\mathrm{Tr}\left(\gamma^\mu\gamma^\nu\right) \\
&= 4\left(g^{\sigma\mu}g^{\nu\rho} - g^{\sigma\nu}g^{\mu\rho} + g^{\sigma\rho}g^{\mu\nu}\right)
\end{aligned}$$

Contracting this identity with p_1, p_2, p_3, p_4 gives the desired result.

10. First,

$$\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right) = -\mathrm{Tr}\left(\gamma^\mu\gamma^5\gamma^\nu\right) = -\mathrm{Tr}\left(\gamma^5\gamma^\nu\gamma^\mu\right),$$

so that $\mathrm{Tr}\left(\gamma^5\gamma^\mu\gamma^\nu\right)$ is skew in $\mu\,\nu$. To calculate this trace we need therefore only look at the cases $\mu = i$, $\nu = 0$ and $\mu = i$, $\nu = j$, $i \neq j$. First, for $\mu = i$, $\nu = 0$,

$$\begin{aligned}
\mathrm{Tr}\left(\gamma^5\gamma^0\gamma^i\right) &= i\mathrm{Tr}\left((\gamma^0)2\gamma^1\gamma^2\gamma^3\gamma^i\right) \\
&= i\mathrm{Tr}\left(\gamma^1\gamma^2\gamma^3\gamma^i\right)
\end{aligned}$$

As γ^i appears somewhere in the product $\gamma^1\gamma^2\gamma^3$, it may be commuted through the relevant factors until we obtain a square $(\gamma^i)2 = -\mathrm{I}$. Thus, up to sign, we have, for $j < k$ and $j, k \neq i$,

$$\mathrm{Tr}\left(\gamma^5\gamma^i\gamma^0\right) = \pm i\mathrm{Tr}\left(\gamma^j\gamma^k\right) = \pm 4i\delta^{jk} = 0,$$

where the result (9) has been used.

When $\mu = i$, $\nu = j$, $i \neq j$, we similarly have, for $k \neq i, j$,

$$\mathrm{Tr}\left(\gamma^5\gamma^i\gamma^j\right) = \pm i\mathrm{Tr}\left(\gamma^0\gamma^k\right) = \pm 2ig^{0k} = 0,$$

again by (9).

11. Again, perform anti-commutations and use the Clifford relations,

$$\begin{aligned}
\gamma_\mu\not{p}\gamma^\mu = \gamma^\nu\gamma^\sigma\gamma^\mu g_{\nu\mu}p_\sigma &= \gamma^\nu\{\gamma^\sigma,\gamma^\mu\}g_{\nu\mu}p_\sigma - \gamma^\nu\gamma^\mu\gamma^\sigma g_{\nu\mu}p_\sigma \\
&= 2\gamma^\nu g^{\sigma\mu}g_{\nu\mu}p_\sigma - g_{\nu\mu}\left(\frac{1}{2}\{\gamma^\nu,\gamma^\mu\} + \frac{1}{2}[\gamma^\nu,\gamma^\mu]\right)\gamma^\sigma p_\sigma \\
&= 2\not{p} - g_{\nu\mu}g^{\nu\mu}\not{p} \\
&= -2\not{p}, \qquad (11)
\end{aligned}$$

where in the third line

$$g_{\nu\mu}[\gamma^\nu,\gamma^\mu] = 0,$$

since $g_{\nu\mu}$ is symmetric $\nu\,\mu$ yet $[\gamma^\nu,\gamma^\mu]$ is skew.

12. Using (11) in the second line,

$$\begin{aligned} \gamma_\mu \not{p}_1 \not{p}_2 \gamma^\mu &= \gamma_\mu \not{p}_1 \{\not{p}_1, \gamma^\mu\} - \gamma_\mu \not{p}_1 \gamma^\mu \not{p}_2 \\ &= 2\gamma_\mu \not{p}_1 p_2^\mu + 2\not{p}_1 \not{p}_2 \\ &= 2\not{p}_2 \not{p}_1 + 2\not{p}_1 \not{p}_2 \\ &= 2\{\not{p}_2, \not{p}_1\} \\ &= 4p \cdot q \end{aligned} \tag{12}$$

13. Using (12),

$$\begin{aligned} \gamma_\mu \not{p}_1 \not{p}_2 \not{p}_3 \gamma^\mu &= \gamma_\mu \not{p}_1 \not{p}_2 \{\not{p}_3, \gamma^\mu\} - \gamma^\mu \not{p}_1 \not{p}_2 \gamma^\mu \not{p}_3 \\ &= 2\not{p}_3 \not{p}_1 \not{p}_2 - 4p_1 \cdot p_2 \not{p}_3 \\ &= 2\not{p}_3 \{\not{p}_1, \not{p}_2\} - 2\not{p}_3 \not{p}_2 \not{p}_1 - 4p_1 \cdot p_2 \not{p}_3 \\ &= -2\not{p}_3 \not{p}_2 \not{p}_1. \end{aligned}$$

14. We show first that

$$S^{\mu\nu\rho\sigma} \equiv \mathrm{Tr}\left(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\right) = \mathrm{Tr}\left(\gamma^5 \gamma^{[\mu} \gamma^\nu \gamma^\rho \gamma^{\sigma]}\right),$$

so that, as a totally skew contravariant 4-tensor in 4 dimensions, $S^{\mu\nu\rho\sigma}$ must be proportional to the alternating tensor $\epsilon^{\mu\nu\rho\sigma}$.

We can show this for adjacent pairs of indices since, for instance,

$$\begin{aligned} \mathrm{Tr}\left(\gamma^5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\right) &= \mathrm{Tr}\left[\gamma^5 \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right) \gamma^\rho \gamma^{si}\right] \\ &= g^{\mu\nu} \mathrm{Tr}\left(\gamma^5 \gamma^\rho \gamma^\sigma\right) + \frac{1}{2} \mathrm{Tr}\left(\gamma^5 [\gamma^\mu, \gamma^\nu] \gamma^\rho \gamma^\sigma\right), \end{aligned}$$

the first term of which vanishes by the identity (10), leaving the term skew in $\mu\,\nu$. This then implies that $S^{\mu\nu\rho\sigma}$ is totally skew since, for instance,

$$S^{\rho\nu\mu\sigma} = -S^{\nu\rho\mu\sigma} = +S^{\nu\mu\rho\sigma} = -S^{\mu\nu\rho\sigma}.$$

Recall that the alternating tensor $\epsilon^{\mu\nu\rho\sigma}$ is defined by

$$\epsilon^{\mu\nu\rho\sigma} = \epsilon_{\alpha\beta\gamma\delta} g^{\alpha\mu} g^{\beta\nu} g^{\gamma\rho} g^{\delta\sigma}$$

where $\epsilon_{0123} = +1$. Therefore

$$\epsilon^{0123} = \epsilon_{\alpha\beta\gamma\delta} g^{\alpha 0} g^{\beta 1} g^{\gamma 2} g^{\delta 3} = \det(g) = -1.$$

Thus

$$\begin{aligned} S^{\mu\nu\rho\sigma} &= -\epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left(\gamma^5 \gamma^0 \gamma^1 \gamma^2 \gamma^3\right) \\ &= i\epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}\left((\gamma^5)2\right) \\ &= i\epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(\mathrm{I}_4) \\ &= 4i\epsilon^{\mu\nu\rho\sigma} \end{aligned}$$

Exercise 4

The Weyl (or chiral) representation of the Clifford algebra is defined by the matrices

$$\gamma^0 = -i \begin{pmatrix} 0 & \mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix}, \ \gamma^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

This may be written more concisely as

$$\gamma^\mu = -i \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (\mathrm{I}_2, \sigma^i)$ and $\bar{\sigma}^\mu = (\mathrm{I}_2, -\sigma^i)$. This is indeed a representation as, using the identities

$$\{\sigma^i, \sigma^j\} = 2\delta^{ij}\,\mathrm{I}_2\ ,\ (\sigma^i)^2 = 1, \tag{13}$$

we see that

$$(\gamma^0)^2 = -\mathrm{I}_4 \quad (\gamma^i)^2 = \begin{pmatrix} (\sigma^i)^2 & 0 \\ 0 & (\sigma^i)^2 \end{pmatrix} = \mathrm{I}_4$$
$$\{\gamma^0, \gamma^i\} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} + \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix} = 0$$
$$\{\gamma^i, \gamma^j\} = \begin{pmatrix} \{\sigma^i, \sigma^j\} & 0 \\ 0 & \{\sigma^i, \sigma^j\} \end{pmatrix} = 2\delta^{ij}\,\mathrm{I}_4$$

You have already met the Dirac representation

$$\tilde{\gamma}^0 = -i \begin{pmatrix} \mathrm{I}_2 & 0 \\ 0 & -\mathrm{I}_2 \end{pmatrix} , \ \tilde{\gamma}^i = -i \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

It is an elementary fact about Clifford algebras that there is only one irreducible representation, and therefore as the above representations have the same dimension they must be equivalent. Let us show this explicitly, i.e. find a matrix U, which we may take to be unitary, such that $\gamma'^\mu = U\gamma^\mu U^\dagger$. Let U be given by

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \ A,\ B,\ C,\ D \in \mathrm{Mat}_{2\times 2}\,\mathbb{C}.$$

Unitarity implies

$$AA^\dagger + BB^\dagger = \mathrm{I}_2 \ , \ CC^\dagger + DD^\dagger = \mathrm{I}_2 \ , \ AC^\dagger + BD^\dagger = \mathrm{I}_2 \tag{14}$$

Now, as $\gamma'^0 U = U\gamma^0$, we then have $A = B$ and $C = -D$, which implies

$$2AA^\dagger = \mathrm{I}_2 \ , \ 2CC^\dagger = \mathrm{I}_2 \ , \ 2BB^\dagger = \mathrm{I}_2 \ , \ 2DD^\dagger = \mathrm{I}_2$$

If we rescale, setting,

$$A = \frac{1}{\sqrt{2}}A', \ C = \frac{1}{\sqrt{2}}C'$$

one finds A' and C' and are unitary, and we can write U (dropping primes for ease of notation) as

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} A & A \\ C & -C \end{pmatrix}$$

Finally, $\gamma'^i U = U\gamma^i$ implies that $A\sigma^i = -\sigma^i C$ and $-\sigma^i A = C\sigma^i$. One solution is given by $A = \mathrm{I}_2$, $C = -\mathrm{I}_2$. Hence U is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

Exercise 5

Begin with the Dirac equation

$$(\gamma^\mu \partial_\mu + m)\,\psi = 0$$

and take the adjoint

$$0 = \partial_\mu \psi^\dagger (\gamma^\mu)^\dagger + m\psi^\dagger.$$

Now, multiply on the right by γ^0,

$$0 = -\partial_\mu \bar{\psi}\left[\gamma^0(\gamma^\mu)^\dagger\gamma^0\right] + m\bar{\psi}$$

Since the Pauli matrices are Hermitian, we find that in the Weyl representation

$$(\gamma^0)^\dagger = -\gamma^0 \ , \ (\gamma^i)^\dagger = \gamma^i.$$

Furthermore, these relations hold in any representation unitarily equivalent to the Weyl representation. Then as $\gamma^0\gamma^i = -\gamma^i\gamma^0$ and $(\gamma^0)^2 = -\mathrm{I}_4$ we have $\gamma^0(\gamma^\mu)^\dagger\gamma^0 = \gamma^\mu$, and we find that the Dirac conjugate $\bar{\psi}$ satisfies

$$\partial_\mu \bar{\psi}\gamma^\mu - m\bar{\psi} = 0.$$

This may be obtained from the Dirac Lagrangian

$$\mathcal{L} = i\bar{\psi}\,(\gamma^\mu \partial_\mu + m)\,\psi$$

as the field equation for ψ

$$\begin{aligned} 0 &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi} \right) - \frac{\partial \mathcal{L}}{\partial \psi} \\ &= \partial_\mu \left(\bar{\psi}\gamma^\mu \right) - m\bar{\psi}. \end{aligned}$$

Recall that the Lagrangian above is real, and therefore the phase change

$$\psi \longmapsto e^{i\alpha}\psi \ , \ \alpha \in \mathbb{R}$$

is a symmetry. In particular, as in Exercise 3 of Example Sheet 1,

$$\Delta\psi = i\psi \ , \ \Delta\bar{\psi} = -i\bar{\psi} \ , \ \Delta\mathcal{L} = 0$$

and so the associated Noether current is

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\Delta\psi + \Delta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\bar{\psi}} \\ &= \bar{\psi}\gamma^\mu\psi \end{aligned}$$

Finally, consider $j^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. Recalling that $\gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu$, we have also $j^\mu = -\bar{\psi}\gamma^5\gamma^\mu\psi$, and so its four-divergence is

$$\begin{aligned} \partial_\mu j^\mu &= (\partial_\mu\bar{\psi}\gamma^\mu)\gamma^5\psi - \bar{\psi}\gamma^5(\gamma^\mu\partial_\mu\psi) \\ &= 2m\bar{\psi}\gamma^5\psi, \end{aligned}$$

where we make use of the equations of motion. We see then that j^μ is conserved when $m = 0$.

Exercise 6

Recall that $(\gamma^5)^2 = 1$ and that, in consequence, the eigenvalues of γ^5 are ± 1. We call a Dirac spinor ψ *left-handed* if $\gamma^5\psi = +\psi$ and *right-handed* if $\gamma^5\psi = -\psi$. The following operators project onto the ± 1 eigenspaces of γ^5,

$$P_\pm := \frac{1}{2}\left(1 \pm \gamma^5\right).$$

We call $\psi_R = P_-\psi$ the right-handed part of ψ and $\psi_L = P_+\psi$ the left-handed part. Note this is a projection operator as $P_\pm^2 = P_\pm$ and $P_+P_- = 0$. Given a Dirac spinor ψ we have the obvious decomposition into left- and right-handed parts:

$$\psi = \psi_L + \psi_R = P_+\psi + P_-\psi. \tag{15}$$

Moreover, such a decomposition is clearly unique for if there exists another decomposition $\psi = P_+\psi' + P_-\psi'$ then

$$P_-\psi = \psi_R = \psi'_R, \tag{16}$$

$$P_+\psi = \psi_L = \psi'_L. \tag{17}$$

Now γ^μ maps left- and right-handed spinors into each other as $\gamma^\mu P_\pm = P_\mp\gamma^\mu$, since γ^5 anti-commutes with γ^μ. Hence

$$\gamma^\mu\psi_L = \gamma^\mu P_+\psi = P_-(\gamma^\mu\psi) = \psi'_R, \tag{18}$$

$$\gamma^\mu\psi_R = \gamma^\mu P_-\psi = P_+(\gamma^\mu\psi) = \psi'_L. \tag{19}$$

If we suppose that ψ satisfies the Dirac equation

$$(\gamma^\mu\partial_\mu + m)\psi = 0,$$

we can project onto the right-handed parts by acting on the left by P_-,

$$0 = \partial_\mu\left(P_-\gamma^\mu\psi\right) + mP_-\psi.$$

Owing to the identity $\{\gamma^5, \gamma^\mu\} = 0$ discussed above, we have $P_+\gamma^\mu = \gamma^\mu P_-$, and so we obtain the following relation amongst the left- and right-handed parts of ψ

$$0 = \gamma^\mu\partial_\mu\psi_L + m\psi_R.$$

An identical calculation produces

$$0 = \gamma^\mu\partial_\mu\psi_R + m\psi_L.$$

If $m \neq 0$, it is therefore not possible for a non-trivial solution to the Dirac equation to have vanishing left- or right-handed parts, else, if, say, $\psi_L = 0$, then

$$0 = \gamma^\mu\partial_\mu\psi_L + m\psi_R = m\psi_R,$$

i.e. $\psi_R = 0$, and so $\psi = \psi_L + \psi_R = 0$.

Suppose now that $m = 0$. To find solutions $\psi = \psi_L + \psi_R$ to the massless field equations, known as the Weyl equations,

$$\gamma^\mu \partial_\mu \psi_L = 0 \ , \ \gamma^\mu \partial_\mu \psi_R = 0, \tag{20}$$

it is necessary to choose a representation for which γ^5 is diagonal,

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Such a choice is the Weyl representation studied in Exercise 1. Indeed, we may have performed the above analysis by specializing to this representation, for which

$$P_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad P_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and the decomposition $\psi = \psi_L + \psi_R$ may be identified with

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}.$$

Although such an approach is perfectly valid, its validity is somewhat limited in that it depends on the choice of representation (and at the level of sophistication involved in this course, we don't really know much about the representations of Clifford algebras) and it is also particular to only 4 space-time dimensions. The ideas above readily generalize to any number $n+1 = d$ space-time dimensions, where S is replaced by a complex vector space of dimension $\lfloor \frac{d}{2} \rfloor$ forming an irreducible representation of the Clifford algebra in $n+1$ space-time dimensions, and for which there is an analogue of γ^5. This is the case precisely when d is even; when d is odd no such γ^5 exists and, consequently, there is no notion of left- or right-handedness of spinors.

Now, to return to the computation at hand, a left-handed spinor is of the form

$$\psi = \begin{pmatrix} \chi \\ 0 \end{pmatrix}.$$

Such solutions to equations (20) thus satisfy

$$i\bar{\sigma}^\mu \partial_\mu \chi = 0.$$

Considering only plane wave solutions,

$$\chi = e^{ip\cdot x} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \ , \ \alpha, \beta \in \mathbb{C},$$

we find

$$0 = (p \cdot \bar{\sigma})\chi = \begin{pmatrix} p_0 - p_3 & p_1 + ip_2 \\ p_1 - ip_2 & p_0 + p_3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -p^0 - p^3 & p^1 + ip^2 \\ p^1 - ip^2 & -p^0 + p^3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

This possesses a solution iff the matrix

$$p \cdot \sigma = \begin{pmatrix} -p^0 - p^3 & p^1 + ip^2 \\ p^1 - ip^2 & -p^0 + p^3 \end{pmatrix}$$

has a non-trivial kernel, i.e. iff

$$0 = \det \begin{pmatrix} -p^0 - p^3 & p^1 + ip^2 \\ p^1 - ip^2 & -p^0 + p^3 \end{pmatrix} = (p^0)^2 - |\mathbf{p}|^2, \tag{21}$$

so that the wave is massless. This is to be expected since the massless Dirac equation implies the massless Klein-Gordon equation,

$$\partial^2 \chi = 0 \ \Rightarrow \ p^2 = 0.$$

so that (21) is satisified.

What, then, are the solutions to

$$0 = \begin{pmatrix} -p^0 - p^3 & p^1 + ip^2 \\ p^1 - ip^2 & -p^0 + p^3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{22}$$

Using the identity of the following exercise, we know that

$$(p \cdot \bar{\sigma})(p \cdot \sigma) = p \cdot p = 0,$$

so that two solutions to (22) are readily obtained,

$$(p\cdot\sigma)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}-p^0+p^3\\p^1+ip^2\end{pmatrix}\quad\text{and}\quad(p\cdot\sigma)\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}p^1-ip^2\\-p^0-p^3\end{pmatrix}.$$

Are these the only solutions and are they independent? The answer to the latter question is no, since if $p^\mu\neq(p^0,0,0,\pm p^0)$, then

$$\begin{pmatrix}-p^0+p^3\\p^1+ip^2\end{pmatrix}=\frac{p^1+ip^2}{-p^0-p^3}\begin{pmatrix}p^1-ip^2\\p^0-p^3\end{pmatrix},$$

this following from the condition $p\cdot p=0$. The momenta $p^\mu=(p^0,0,0,\pm p^0)$ are excluded as one of these solutions vanishes in this case.

Now, that these are, up to proportionality, the only solutions follows from a rather simple observation: if there were two independent solutions then we'd have

$$\dim\ker(p\cdot\bar\sigma)=2,$$

which, since $\dim S_L=2$, would imply that $(p\cdot\bar\sigma)$ were identically zero. However, this occurs if and only if $p^\mu=0$, giving constant χ. Excluding such trivialities, we see then that, for any given 4-momentum $p^\mu\neq0$, the space of solutions is precisely 1 dimensional.

If, instead, only the 3-momentum $\mathbf{p}$ is specified, there are two independent solutions, corresponding to plane waves with $p^\mu=(\pm|\mathbf{p}|,\mathbf{p})$, i.e. positive and negative frequency waves.

Exercise 7

Recall the definitions

$$(i\not{p}+m)u_r(p)=0\ ,\ (-i\not{p}+m)v_r(p)=0\tag{23}$$

together with the orthogonality relations

$$\bar u_r(p)u_s(p)=-2im\delta_{rs}\ ,\ \bar v_r(p)v_s(p)=2im\delta_{rs}\ ,\ \bar u_r(p)v_s(p)=\bar v_r(p)u_s(p)=0\tag{24}$$

for any r,s and p. The latter tell us that the set $\left\{u_{\pm\frac12}(p),v_{\pm\frac12}(p)\right\}$ is a basis of the four dimensional vector space of Dirac spinors for every fixed value of p. The linear operators

$$A(p):=\sum_{s=\pm\frac12}u_s(p)\bar u_s(p)\ ,\ B(p):=\sum_{s=\pm\frac12}v_s(p)\bar v_s(p)$$

are therefore determined entirely by their action on the basis spinors $u_r(p)$ and $v_r(p)$. We have then

$$\begin{aligned}A(p)u_r(p)&=\sum_s u_s(p)\left[\bar u_s(p)u_r(p)\right]\\&=-2im\sum_s u_s(p)\delta_{rs}\\&=-2im\,u_r(p)\\&=(-\not{p}-im)u_r(p)\end{aligned}$$

where we use the first orthogonality relation in (24) and, in the final step, the first equation in (23) to evaluate

$$-2im\,u_r(p)=-im\,u_r(p)-im\,u_r(p)=-\not{p}u_r(p)-im\,u_r(p).$$

Similarly,

$$\begin{aligned}A(p)v_r(p)&=\sum_s u_s(p)\left[\bar u_s(p)v_r(p)\right]\\&=0\\&=(-\not{p}-im)\,v_r(p)\end{aligned}$$

It follows then that

$$A(p)=-\not{p}-im.$$

Similarly, evaluate $B(p)$ on the spinors $u_r(p)$ and $v_r(p)$,

$$\begin{aligned}B(p)u_r(p)&=0\\&=(-\not{p}+im)\,u_r(p)\\B(p)v_r(p)&=2im\,v_r(p)\\&=(-\not{p}+im)\,v_r(p)\end{aligned}$$

Exercise 8

Using the expansions given on the problem sheet and dropping vanishing anti-commutators we get

$$\begin{aligned}
\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} &= \sum_{s,r=1}^{2} \int \frac{d^3p d^3q}{(2\pi)^6 2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \left[u_s(\vec{p}) u_r^\dagger(\vec{q}) e^{i(\vec{p}\vec{x}-\vec{q}\vec{y})} \{a^s_{\vec{p}}, a^{r\dagger}_{\vec{q}}\} + v_s(\vec{p}) v_r^\dagger(\vec{q}) e^{-i(\vec{p}\vec{x}-\vec{q}\vec{y})} \{b^{s\dagger}_{\vec{p}}, b^r_{\vec{q}}\} \right] && (25) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left[(\not{p}+im)\gamma^0 e^{i\vec{p}(\vec{x}-\vec{y})} + (\not{p}-im)\gamma^0 e^{-i\vec{p}(\vec{x}-\vec{y})} \right]_{p^0=E_{\vec{p}}}, && (26)
\end{aligned}$$

We got to the second line by first using the anti-commutation relations and then the results from Exercise 7 for the spin sums:

$$\begin{aligned}
\sum_s u_s(\vec{p}) u_s^\dagger(\vec{p}) &= -\sum_s u_s(\vec{p}) \bar{u}_s(\vec{p}) \gamma^0 = (\not{p}+im)\gamma^0 \,, && (27) \\
\sum_s v_s(\vec{p}) v_s^\dagger(\vec{p}) &= -\sum_s v_s(\vec{p}) \bar{v}_s(\vec{p}) \gamma^0 = (\not{p}-im)\gamma^0 . && (28)
\end{aligned}$$

To simplify Eq. (26) we substitute $\vec{p} \to -\vec{p}$ in the second term in the square brackets and write out $\not{p} = p_0\gamma^0 + p_i\gamma^i$:

$$\begin{aligned}
\{\psi(\vec{x}), \psi^\dagger(\vec{y})\} &= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \left[(p_0\gamma^0 + p_i\gamma^i + m + p_0\gamma^0 - p_i\gamma^i - m)\gamma^0 e^{i\vec{p}(\vec{x}-\vec{y})} \right]_{p^0=E_{\vec{p}}} && (29) \\
&= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} - 2p_0 e^{i\vec{p}(\vec{x}-\vec{y})} \mathbf{1}_{4\times 4} = \delta^{(3)}(\vec{x}-\vec{y}) \mathbf{1}_{4\times 4} . && (30)
\end{aligned}$$

Similarly one can show $\{\psi(\vec{x}), \psi(\vec{y})\} = 0 = \{\psi^\dagger(\vec{x}), \psi^\dagger(\vec{y})\}$. Note that ψ's are Dirac spinors with 4 components ψ_α, $\alpha = 1,2,3,4$ and therefore we computed a 4×4 matrix. We can write out the spinor indices as

$$\{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}). \tag{31}$$

Exercise 9

We want to express the Hamiltonian, given by:

$$H = i \int d^3x \; \bar{\psi} \left(\gamma^i \partial_i + m\right) \psi \,, \tag{32}$$

in terms the the creation and annihilation operators for the Dirac fields. First begin by using the equation of motion

$$\left(\gamma^a \partial_a + m\right) \psi = 0 \,, \tag{33}$$

to rewrite

$$\left(\gamma^i \partial_i + m\right) \psi = -\gamma^0 \partial_0 \psi \,, \tag{34}$$

and hence

$$H = -i \int d^3x \; \bar{\psi} \gamma^0 \partial_0 \psi \,. \tag{35}$$

We now need to plug in the expansions (6) from the problem sheet, and compute. First of all,

$$-\gamma^0\partial_0\psi = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \left(-i\gamma^0 q_0\right) \sum_{r=1}^{2} \left[a^r_{\vec{q}} u_r(\vec{q}) e^{i\vec{q}\cdot\vec{x}} - b^{r\dagger}_{\vec{q}} v_r(\vec{q}) e^{-i\vec{q}\cdot\vec{x}} \right] \tag{36}$$

and so

$$H = -\int d^3x \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{q}}}} \sum_{r=1}^{2} q_0 \psi^\dagger \left(a^r_{\vec{q}} u_r(\vec{q}) e^{i\vec{q}\cdot\vec{x}} - b^{r\dagger}_{\vec{q}} v_r(\vec{q}) e^{-i\vec{q}\cdot\vec{x}} \right) , \tag{37}$$

Now, we have no choice but to plug in the expression for $\bar{\psi}$ in terms of b^s, $a^{s\dagger}$ and expand

$$\begin{aligned}
H = -\int d^3x \int\int \frac{d^3p\; d^3q}{(2\pi)^6} \frac{q_0}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \sum_{s,r} \Big[& a^{s\dagger}_{\vec{p}} a^r_{\vec{q}} u_s^\dagger(\vec{p}) u_r(\vec{q}) e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} + b^s_{\vec{p}}\, a^r_{\vec{q}} v_s^\dagger(\vec{p}) u_r(\vec{q}) e^{i(\vec{p}+\vec{q})\cdot\vec{x}} \\
& - a^{s\dagger}_{\vec{p}} b^{r\dagger}_{\vec{q}} u_s^\dagger(\vec{p}) v_r(\vec{q}) e^{-i(\vec{p}+\vec{q})\cdot\vec{x}} - b^s_{\vec{p}}\, b^{r\dagger}_{\vec{q}} v_s^\dagger(\vec{p}) v_r(\vec{q}) e^{i(\vec{p}-\vec{q})\cdot\vec{x}} \Big] .
\end{aligned}$$

The spatial integral gives δ-functions in momentum space

$$H = -\int\int \frac{d^3p\ d^3q}{(2\pi)^3} \frac{q_0}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \sum_{s,r} \Big[a^{s\dagger}_{\vec{p}} a^r_{\vec{q}} u^\dagger_s(\vec{p}) u_r(\vec{q}) \delta^{(3)}(\vec{p}-\vec{q}) + b^s_{\vec{p}}\ a^r_{\vec{q}} v^\dagger_s(\vec{p}) u_r(\vec{q}) \delta^{(3)}(\vec{p}+\vec{q})$$
$$-a^{s\dagger}_{\vec{p}} b^{r\dagger}_{\vec{q}} u^\dagger_s(\vec{p}) v_r(\vec{q}) \delta^{(3)}(\vec{p}+\vec{q}) - b^s_{\vec{p}}\ b^{r\dagger}_{\vec{q}} v^\dagger_s(\vec{p}) v_r(\vec{q}) \delta^{(3)}(\vec{p}-\vec{q}) \Big]$$
$$= -\int \frac{d^3p}{(2\pi)^3} \frac{p_0}{2E_{\vec{p}}} \sum_{s,r} \Big[a^{s\dagger}_{\vec{p}} a^r_{\vec{p}} u^\dagger_s(\vec{p}) u_r(\vec{p}) + b^s_{\vec{p}}\ a^r_{-\vec{p}} v^\dagger_s(\vec{p}) u_r(-\vec{p})$$
$$-a^{s\dagger}_{\vec{p}} b^{r\dagger}_{-\vec{p}} u^\dagger_s(\vec{p}) v_r(-\vec{p}) - b^s_{\vec{p}}\ b^{r\dagger}_{\vec{p}} v^\dagger_s(\vec{p}) v_r(\vec{p}) \Big]$$
$$= \int \frac{d^3p}{(2\pi)^3} \sum_{s,r} \Big[a^{s\dagger}_{\vec{p}} a^r_{\vec{p}} u^\dagger_s(\vec{p}) u_r(\vec{p}) + b^s_{\vec{p}}\ a^r_{-\vec{p}} v^\dagger_s(\vec{p}) u_r(-\vec{p})$$
$$-a^{s\dagger}_{\vec{p}} b^{r\dagger}_{-\vec{p}} u^\dagger_s(\vec{p}) v_r(-\vec{p}) - b^s_{\vec{p}}\ b^{r\dagger}_{\vec{p}} v^\dagger_s(\vec{p}) v_r(\vec{p}) \Big]\ ,$$

where in the last line we used the fact that $p_0 = -E_p$. The dot products $u^\dagger_s \cdot v_r$ etc can be calculated from the explicit form of the wave functions given in lectures. This gives

$$u^\dagger_s(\vec{p}) \cdot u_r(\vec{q}) = 2\sqrt{p^0 q^0}\delta_{rs}\,, \qquad v^\dagger_s(\vec{p}) \cdot v_r(\vec{q}) = 2\sqrt{p^0 q^0}\delta_{rs}\,, \qquad u^\dagger_s(\vec{p}) v_r(\vec{q}) = v^\dagger_s(\vec{p}) u_r(\vec{q}) = 0\,. \tag{38}$$

Using these identities one finds

$$\begin{aligned} H &= \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \sum_s \Big[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} - b^s_{\vec{p}}\ b^{s\dagger}_{\vec{p}} \Big] \\ &= \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \Big[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} - (2\pi)^3 \delta^{(3)}(0) \Big]\ . \end{aligned} \tag{39}$$

Upon normal ordering, we obtain

$$:H: = \sum_s \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \Big[a^{s\dagger}_{\vec{p}} a^s_{\vec{p}} + b^{s\dagger}_{\vec{p}} b^s_{\vec{p}} \Big]\ . \tag{40}$$

Exercise 10

We will quantise a spin 1/2 Dirac field with boson commutation relations, which is wrong because you need anti-commutation relations for fermions. The discussion here closely follows chapter 5.1 in Tong's notes.
First we expand $\psi(\vec{x})$ and $\psi^\dagger(\vec{x})$ with creation and annihilation operators as in Eq. (6) on the problem sheet. Then we impose boson commutation relations

$$[a^r_{\vec{p}}, a^{s\dagger}_{\vec{q}}] = (2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q}), \qquad [b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}] = -(2\pi)^3 \delta^{rs} \delta^{(3)}(\vec{p}-\vec{q}) \tag{41}$$

with all other commutators vanishing. Now we can repeat exercise 8 with commutation instead of anti-commutation relations. Using the expansions of ψ and $\psi^\dagger$ in (6) on the problem sheet and only writing out non-vanishing commutators we find

$$[\psi(\vec{x}), \psi^\dagger(\vec{y})] = \sum_{s,r=1}^{2} \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}} \Big[u_s(\vec{p}) u^\dagger_r(\vec{q}) e^{i(\vec{p}\vec{x}-\vec{q}\vec{y})} [a^s_{\vec{p}}, a^{r\dagger}_{\vec{q}}] + v_s(\vec{p}) v^\dagger_r(\vec{q}) e^{-i(\vec{p}\vec{x}-\vec{q}\vec{y})} [b^{s\dagger}_{\vec{p}}, b^r_{\vec{q}}] \Big] \tag{42}$$

$$= \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} \Big[(\not{p} + im)\gamma^0 e^{i\vec{p}(\vec{x}-\vec{y})} + (\not{p} - im)\gamma^0 e^{-i\vec{p}(\vec{x}-\vec{y})} \Big]_{p^0 = E_{\vec{p}}}. \tag{43}$$

This is the same result as in Eqs. (27)-(28) in exercise 8. We showed in exercise 8 that the last expression can be simplified to give $\delta^{(3)}(\vec{x}-\vec{y})\mathbf{1}_{4\times 4}$. Therefore

$$[\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})] = \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}), \tag{44}$$

where $\alpha, \beta \in \{1,2,3,4\}$ are spinor indices. We also find $[\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] = 0 = [\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})]$.
Let us compute the Dirac Hamiltonian

$$H = i\int d^3x \bar{\psi}(\gamma^i \partial_i + m)\psi. \tag{45}$$

As in exercise 9, our strategy will be to use the expansion of $\psi, \psi^\dagger$ in terms of creation and annihilation operators from Eq. (6) on the problem sheet and then simplify the result. Since we expect a simplification similar to what happened in exercise 9 to happen again, we start by looking at:

$$(\gamma^i\partial_i + m)\psi = \sum_{s=1}^{2} \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_{\vec{p}}}} \Big[(\gamma^i(ip_i) + m) u_s(\vec{p}) a^s_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + (\gamma^i(-ip_i) + m) v_s(\vec{p}) v^{s\dagger}_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}} \Big] \tag{46}$$

We simplify this using

$$(i\not{p}+m)u_s(\vec{p})=0 \qquad \Rightarrow \qquad ip_j\gamma^j u_s(\vec{p})=(ip_0\gamma^0-m)u_s(\vec{p}) \tag{47}$$
$$(-i\not{p}+m)v_s(\vec{p})=0 \qquad \Rightarrow \qquad -ip_j\gamma^j v_s(\vec{p})=(-ip_0\gamma^0-m)v_s(\vec{p}) \tag{48}$$

so once again the masses cancel out and we are left with a term proportional to $p_0\gamma^0 u_s(\vec{p})$ and one proportional to $-p_0\gamma^0 v_s(\vec{p})$ in Eq. (46). Noting that $p_0=-E_{\vec{p}}$ (since $p^0=E_{\vec{p}}$) and proceeding to the change of variables $\vec{p}\to-\vec{p}$ in the v_s term, we get

$$(\gamma^i\partial_i+m)\psi=\sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\sqrt{\frac{E_{\vec{p}}}{2}}\,(-i\gamma^0)\left[u_s(\vec{p})a^s_{\vec{p}}-v_s(-\vec{p})b^{s\dagger}_{-\vec{p}}\right]e^{i\vec{p}\cdot\vec{x}}. \tag{49}$$

Therefore the full Hamiltonian becomes

$$\begin{aligned}
H &= i\int d^3x(\psi^\dagger\gamma^0)(\gamma^i\partial_i+m)\psi && (50)\\
&= i\sum_{s,r=1}^{2}\int d^3x\int\frac{d^3pd^3q}{(2\pi)^6}\sqrt{\frac{E_{\vec{p}}}{2}}\frac{i}{\sqrt{2E_{\vec{q}}}}\left[a^{r\dagger}_{\vec{q}}u^\dagger_r(\vec{q})+b^r_{-\vec{q}}v^\dagger_r(-\vec{q})\right]\left[a^s_{\vec{p}}u_s(\vec{p})-b^{s\dagger}_{-\vec{p}}v_s(-\vec{p})\right]e^{i\vec{x}\cdot(\vec{p}-\vec{q})} && (51)\\
&= i\sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{-i}{2}2E_{\vec{p}}\left(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}}-b^s_{\vec{p}}b^{s\dagger}_{\vec{p}}\right) && (52)\\
&= \sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}E_{\vec{p}}\left(a^{s\dagger}_{\vec{p}}a^s_{\vec{p}}-b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}+(2\pi)^3\delta^{(3)}(0)\right). && (53)
\end{aligned}$$

We remove the Dirac delta function in the last line by normal ordering. The $-b^{s\dagger}_{\vec{p}}b^s_{\vec{p}}$ term means that we can reduce the energy by creating b particles, i.e. the energy is unbounded below. Because of the wrong commutation relations we are using, even if we were to redefine $b^s_{\vec{p}}\to b^{s\dagger}_{\vec{p}}$, $b^{s\dagger}_{\vec{p}}\to b^s_{\vec{p}}$, and then re-do the normal-ordering procedure, the problematic term would be negative, which means there is no redefinition we can do to make it bounded from bellow. (Note that if we had been using anti-commutation relations and had gotten this result, this redefinition would have done the trick, so this is truly a consequence of the wrong commutation relations that we were using.)

Exercise 11

We want to compute the Feynman propagator of the Dirac field, which is defined as

$$S_F(x-y)\equiv\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle\equiv\begin{cases}\langle 0|\psi(x)\bar{\psi}(y)|0\rangle & \text{for } x^0>y^0,\\ -\langle 0|\bar{\psi}(y)\psi(x)|0\rangle & \text{for } x^0<y^0.\end{cases} \tag{54}$$

In the case $x^0>y^0$ we have

$$\begin{aligned}
\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \sum_{s,r=1}^{2}\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\langle 0|\left(a^s_{\vec{p}}u_s(\vec{p})e^{ipx}+b^{s\dagger}_{\vec{p}}v_s(\vec{p})e^{-ipx}\right)\left(a^{r\dagger}_{\vec{q}}\bar{u}_r(\vec{q})e^{-iqy}+b^r_{\vec{q}}\bar{v}_r(\vec{q})e^{iqy}\right)|0\rangle && (55)\\
&= \sum_{s,r=1}^{2}\int\frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\langle 0|a^s_{\vec{p}}a^{r\dagger}_{\vec{q}}u_s(\vec{p})\bar{u}_r(\vec{q})e^{i(px-qy)}|0\rangle. && (56)
\end{aligned}$$

Now, performing the anti-commutation $a^s_{\vec{p}}a^{r\dagger}_{\vec{q}}=-a^{r\dagger}_{\vec{q}}a^s_{\vec{p}}+(2\pi)^3\delta_{rs}\delta^{(3)}(\vec{p}-\vec{q})$ and performing the resulting delta function, we have, using the first spin sum identity proven in exercise 7,

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle=\sum_{s=1}^{2}\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}u_s(\vec{p})\bar{u}_s(\vec{p})e^{ip(x-y)}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(-\not{p}-im)e^{ip(x-y)}. \tag{57}$$

Writing out the spinor indices α,β this gives (in the case $x^0>y^0$)

$$S_F(x-y)_{\alpha\beta}=\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}(-p_a(\gamma^a)_{\alpha\beta}-im\delta_{\alpha\beta})e^{ip(x-y)}. \tag{58}$$

The case $x^0<y^0$ works similarly,

$$\begin{aligned}
S_F(x-y)_{\alpha\beta} &= -\langle 0|\bar{\psi}(y)_\beta\psi(x)_\alpha|0\rangle && (59)\\
&= -\sum_{s,r=1}^{2}\int\frac{d^3pd^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\langle 0|\left(a^{s\dagger}_{\vec{p}}\bar{u}_s(\vec{p})_\beta e^{-ipy}+b^s_{\vec{p}}\bar{v}_s(\vec{p})_\beta e^{ipy}\right)\left(a^r_{\vec{q}}u_r(\vec{q})_\alpha e^{iqx}+b^{r\dagger}_{\vec{q}}v_r(\vec{q})_\alpha e^{-iqx}\right)|0\rangle && (60)\\
&= -\sum_{s,r=1}^{2}\int\frac{d^3pd^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\langle 0|b^s_{\vec{p}}b^{r\dagger}_{\vec{q}}\bar{v}_s(\vec{p})_\beta v_r(\vec{q})_\alpha e^{i(py-qx)}|0\rangle. && (61)
\end{aligned}$$

Now, anti-commuting and using the second spin sum proven in exercise 7,

$$(S_F(x-y))_{\alpha\beta} = -\int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(\sum_{s=1}^{2} v_s(\vec{p})\bar{v}_s(\vec{p})\right)_{\alpha\beta} e^{-ip(x-y)} = -\int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}}(-\not{p}+im)_{\alpha\beta}\, e^{-ip(x-y)}. \tag{62}$$

Finally we write both cases $x^0 > y^0$ and $x^0 < y^0$ in one formula as

$$(S_F(x-y))_{\alpha\beta} = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left[\theta(x^0-y^0)\,(-\not{p}-im)_{\alpha\beta}\, e^{ip(x-y)} - \theta(y^0-x^0)\,(-\not{p}+im)_{\alpha\beta}\, e^{-ip(x-y)}\right]. \tag{63}$$

Having obtained this expression for the Feynman propagator, we can use complex analysis techniques to see how it can be recovered from a 4-momentum integral of the form:

$$\int \frac{d^4p}{(2\pi)^4}\frac{(i\not{p}-m)}{p^2+m^2-i\epsilon}e^{ip(x-y)} \tag{64}$$

upon choosing a particular prescription for avoiding the two poles of the integrand occurring on the mass-shell $p^2 = -m^2$ in momentum space, or, in p^0 space, $p^0 = \pm E_{\vec{p}}$
Unlike the computation for the retarded propagator of a scalar field, the Feynman propagator here contains two terms of which only one is non-zero according to the time ordering of the fields. Following the same logic used in the case of the scalar field propagator, we thus move one of the poles into the upper plane so as to be caught inside the contour for $x^0 < y^0$, while moving the other down into the lower half plane so as to be caught inside the semi-circular contour employed in the case $x^0 > y^0$.
This regularization is achieved by making the replacement:

$$\frac{1}{p^2+m^2} \to \frac{1}{p^2+m^2-i\epsilon} = \frac{-1}{\left(p^0+\sqrt{E_{\vec{p}}^2-i\epsilon}\right)\left(p^0-\sqrt{E_{\vec{p}}^2-i\epsilon}\right)}, \tag{65}$$

which, in the limit $\epsilon \to 0$ has simple poles at

$$p^0 = \pm(E_{\vec{p}} - i\frac{\epsilon}{2E_{\vec{p}}}) + \mathcal{O}(\epsilon^2), \tag{66}$$

i.e. $E_{\vec{p}}$ is perturbed downwards while $-E_{\vec{p}}$ is perturbed upwards (see figure). That this gives the right result for S_F we ow demonstrate.

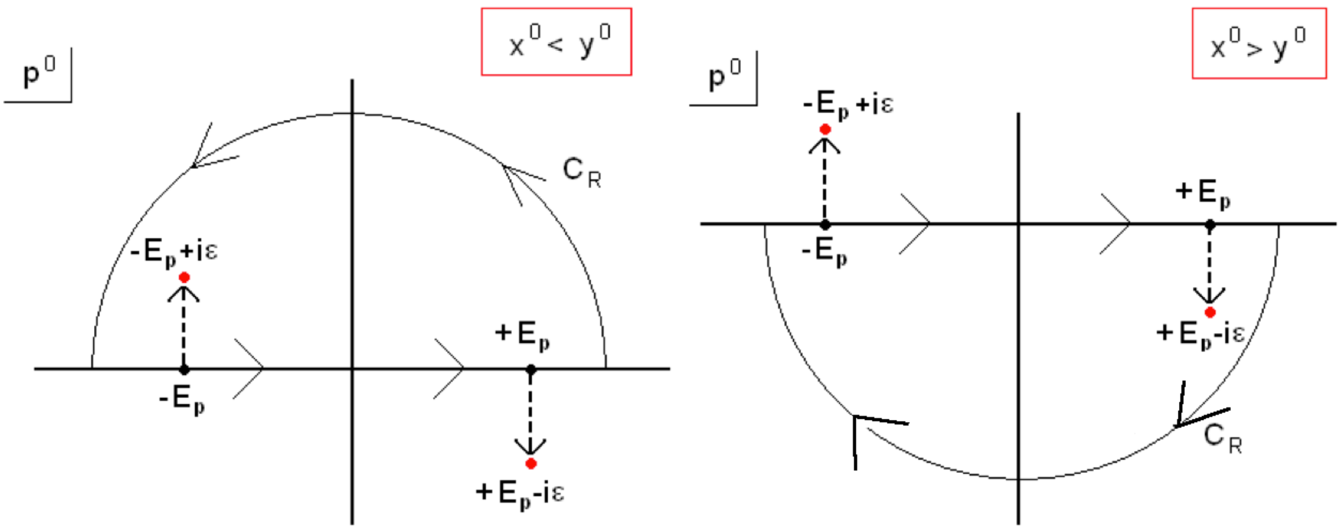

Figure 1: The pole prescription for evaluating the Feynman propagator.

The residue at each of these poles in the above limit are:

$$\lim_{\epsilon\to 0}\mathrm{Res}\left(\frac{-1}{(p^0)^2 - E_{\vec{p}}^2 + i\epsilon}e^{-ip^0(x^0-y^0)},\; p^0 = \pm\sqrt{E_{\vec{p}}^2 - i\epsilon}\right) = \mp\frac{1}{2E_p}e^{\mp iE_{\vec{p}}(x^0-y^0)} \tag{67}$$

Therefore, when $x^0 > y^0$, employing Jordan's lemma and the residue theorem as before, and remembering that there is an additional factor of -1 form the clockwise direction of the contour, we have:

$$\begin{aligned}
\lim_{\epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{(i\not{p}-m)}{p^2+m^2-i\epsilon}e^{ip(x-y)} &= (-2\pi i)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi}\frac{(i\not{p}-m)}{-2E_{\vec{p}}}e^{-iE_{\vec{p}}(x^0-y^0)}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\Big|_{p^0=E_{\vec{p}}} &(68)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{(-\not{p}-im)}{2E_{\vec{p}}}e^{ip(x-y)}\Big|_{p^0=E_{\vec{p}}}\,, &(69)
\end{aligned}$$

which coincides with $S_F(x-y)$ for $x^0 > y^0$.
For $x^0 < y^0$ we instead close in the upper-half plane and obtain

$$\begin{aligned}
\lim_{\epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{(i\not{p}-m)}{p^2+m^2-i\epsilon}e^{ip(x-y)} &= (2\pi i)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi}\frac{(i\not{p}-m)}{2E_{\vec{p}}}e^{iE_{\vec{p}}(x^0-y^0)}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\Big|_{p^0=-E_{\vec{p}}} &(70)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{(-\not{p}-im)}{2E_{\vec{p}}}e^{iE_{\vec{p}}(x^0-y^0)}e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\Big|_{p^0=-E_{\vec{p}}} &(71)\\
&= \int \frac{d^3p'}{(2\pi)^3}\frac{(\not{p}\,'-im)}{2E_{\vec{p}'}}e^{iE_{\vec{p}'}(x^0-y^0)}e^{-i\vec{p}'\cdot(\vec{x}-\vec{y})}\Big|_{p'^0=-E_{\vec{p}'}}\,, &(72)
\end{aligned}$$

where, in the last line, we have changed variables $p^i \to p'^i = -p^i$ for all three spatial components. We made this change because in the line before, neither the $\not{p}$ nor the exponent were explicitly Lorentz scalars when $p^0 = -E_{\vec{p}}$. With this change of variables, we obtain:

$$\begin{aligned}
\not{p}\,|_{p^0=-E_{\vec{p}}} &= \left[-p^0\gamma^0 + p^i\gamma^i\right]\big|_{p^0=-E_{\vec{p}}} &(73)\\
&= E_{\vec{p}}\,\gamma^0 + p^i\gamma^i &(74)\\
&= -\not{p}\,'\,, &(75)
\end{aligned}$$

We finally obtain

$$\lim_{\epsilon\to 0}\int \frac{d^4p}{(2\pi)^4}\frac{(i\not{p}-m)}{p^2+m^2-i\epsilon}e^{ip(x-y)} = -\int \frac{d^3p'}{(2\pi)^3}\frac{(-\not{p}\,'+im)}{2E_{\vec{p}'}}e^{-ip'(x-y)}\Big|_{p'^0=-E_{\vec{p}'}}\,, \tag{76}$$

and this is precisely the expression for $S_F(x-y)$ when $y^0 > x^0$.
Finally we verify that S_F is a Green's function for the Dirac operator:

$$\begin{aligned}
(\not{\partial}_x + m)S_F(x-y) &= (\not{\partial}_x + m)\int \frac{d^4p}{(2\pi)^4}e^{ip(x-y)}\frac{i\not{p}-m}{p^2+m^2} &(77)\\
&= \int \frac{d^4p}{(2\pi)^4}(i\not{p}+m)\frac{i\not{p}-m}{p^2+m^2}e^{ip(x-y)} &(78)\\
&= \int \frac{d^4p}{(2\pi)^4}\frac{-\not{p}^2-m^2}{p^2+m^2}e^{ip(x-y)} &(79)\\
&= -\delta^{(4)}(x-y). &(80)
\end{aligned}$$

Here we used

$$\not{p}^2 = \gamma^a p_a \gamma^b p_b = p_a p_b \frac{1}{2}\left\{\gamma^a, \gamma^b\right\} = p^2\,. \tag{81}$$

Exercise 12

In Maxwell Theory, the dynamics of a co-vector field $A_\mu(x)$ are governed by the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F_{\sigma\tau}\eta^{\mu\sigma}\eta^{\nu\tau} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

in terms of the field-strength (or Faraday) tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.$$

Under the 'gauge' transformation

$$A_\mu \longmapsto A_\mu + \partial_\mu\xi, \tag{82}$$

for ξ any smooth function, the field-strength changes according to

$$\begin{aligned}
F_{\mu\nu} \longmapsto \partial_\mu\left(A_\nu + \partial_\nu\xi\right) - \partial_\nu\left(A_\mu + \partial_\mu\xi\right) &= \partial_\mu A_\nu - \partial_\nu A_\mu + \partial^2_{\mu\nu}\xi - \partial^2_{\nu\mu}\xi\\
&= F_{\mu\nu}
\end{aligned}$$

Where in the last line we employed the fact that mixed partial derivatives commute. We see then that under (82), the field-strength, and therefore the Lagrangian, is unchanged - i.e. *gauge invariant.*

Now, for space-time translations $x^\mu \mapsto x^\mu + a^\mu$, the field transforms by Taylor's Theorem as

$$A_\mu(x) \longmapsto A_\mu + a^\nu \partial_\nu A_\mu(x) + \mathcal{O}(a^2),$$

ie. $\delta A_\mu = a^\nu \Delta_\nu A_\mu$, where $\Delta_\nu A_\mu = \partial_\nu A_\mu$. For the field-strength,

$$F_{\mu\nu} \longmapsto F_{\mu\nu} + a^\lambda \partial^2_{\mu\lambda} A_\nu - a^\lambda \partial^2_{\nu\lambda} A_\mu + \mathcal{O}(a^2) = F_{\mu\nu} + \partial_\lambda \left(a^\lambda F_{\mu\nu}\right) + \mathcal{O}(a^2),$$

with everything evaluated at the space-time point x^μ. Putting this transformation into $\mathcal{L}$ above, we see that the first order change in the Lagrangian is

$$\mathcal{L} \longmapsto \mathcal{L} - \frac{1}{2}\partial_\lambda \left(a^\lambda F_{\mu\nu}\right) F_{\sigma\tau}\eta^{\mu\sigma}\eta^{\nu\tau} = \mathcal{L} - \frac{1}{4}\partial_\lambda \left(a^\lambda F_{\mu\nu}F^{\mu\nu}\right)$$

or $\delta\mathcal{L} = \partial_\lambda(a^\lambda \mathcal{L})$, a divergence.

By Noether's theorem, we obtain conserved currents $T^\mu_{\ \nu}$ for each space-time direction given by the formula (Schroeder & Peskin p. 18)

$$T^\mu_{\ \nu} \quad = \quad \frac{\partial \mathcal{L}}{\partial\, \partial_\mu A_\lambda}\Delta_\nu A_\lambda - \mathcal{J}^\mu_\nu, \tag{83}$$

where, in this instance, $\mathcal{J}^\mu_\nu = \delta^\mu_\nu \mathcal{L}$. Calculate that

$$\frac{\partial\, F_{\sigma\tau}}{\partial\, \partial_\mu A_\nu} = \delta^\mu_\sigma \delta^\nu_\tau - \delta^\nu_\sigma \delta^\mu_\tau$$

so that

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial\, \partial_\mu A_\nu} \quad &= \quad -\frac{1}{2}\frac{\partial\, F_{\sigma\tau}}{\partial\, \partial_\mu A_\nu}F^{\sigma\tau} \\
&= \quad -\frac{1}{2}\left(\delta^\mu_\sigma \delta^\nu_\tau - \delta^\nu_\sigma \delta^\mu_\tau\right)F^{\sigma\tau} \\
&= \quad -\frac{1}{2}\left(F^{\mu\nu} - F^{\nu\mu}\right) \\
&= \quad F^{\nu\mu},
\end{aligned} \tag{84}$$

by skew-symmetry of the field-strength (i.e. $F_{\mu\nu} = -F_{\nu\mu}$). Altogether, the array $T^\mu_{\ \nu}$, or *Energy-Momentum Tensor*, is given by

$$T^\mu_{\ \nu} = F^{\lambda\mu}\partial_\nu A_\lambda + \frac{1}{4}\delta^\mu_\nu F^{\sigma\tau}F_{\sigma\tau}.$$

Let us raise the ν index,

$$T^{\mu\nu} = F^{\lambda\mu}\partial^\nu A_\lambda + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau}.$$

Manifestly, this quantity is not a symmetric tensor and cannot be gauge invariant, since under (82)

$$F^{\lambda\mu}\partial^\nu A_\lambda \longmapsto F^{\lambda\mu}\partial^\nu A_\lambda + F^{\lambda\mu}\partial^\nu\partial_\lambda \xi,$$

the second term of which does not, in general, vanish; very undesirable properties for an energy-momentum distribution.

Instead, let us examine the following tensor

$$\begin{aligned}
\Theta^{\mu\nu} \quad &= \quad T^{\mu\nu} - F^{\lambda\mu}\partial_\lambda A^\nu \\
&= \quad F^{\lambda\mu}\partial^\nu A_\lambda - F^{\lambda\mu}\partial_\lambda A^\nu + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \\
&= \quad F^{\lambda\mu}\left(\partial^\nu A_\lambda - \partial_\lambda A_\nu\right) + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \\
&= \quad F^{\lambda\mu}F_{\sigma\lambda}\eta^{\sigma\nu} + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \\
&= \quad F^{\lambda\mu}F^\nu_{\ \lambda} + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau}
\end{aligned}$$

which manifestly is gauge invariant. Furthermore

$$\begin{aligned}
\Theta^{\nu\mu} &= F^{\lambda\nu}F^{\mu}_{\ \lambda} + \frac{1}{4}\eta^{\nu\mu}F^{\sigma\tau}F_{\sigma\tau} \\
&= -F^{\nu\lambda}F^{\mu}_{\ \lambda} + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \\
&= -F^{\nu}_{\ \lambda}F^{\mu\lambda} + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \\
&= F^{\nu}_{\ \lambda}F^{\lambda\mu} + \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau}
\end{aligned}$$

where we interchange indices $\nu \leftrightarrow \lambda$ in the second line, picking up one factor of (-1), simultaneously raise/lower λ in the third and swap indices again in the final line.

This new tensor is traceless since

$$\begin{aligned}
\Theta^{\mu}_{\ \mu} &= F^{\lambda\mu}F_{\mu\lambda} + \frac{1}{4}\delta^{\mu}_{\mu}F^{\sigma\tau}F_{\sigma\tau} \\
&= -F^{\mu\lambda}F_{\mu\lambda} + F^{\sigma\tau}F_{\sigma\tau} \\
&= 0.
\end{aligned}$$

Finally, we show that $\Theta^{\mu\nu}$ is also conserved. However, a Noether current is only conserved when the fields are 'on-shell,' i.e. satisfy the field equations, which we must therefore calculate:

$$\begin{aligned}
0 &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu A_\nu}\right) - \frac{\partial\mathcal{L}}{\partial A_\nu} \\
&= \partial_\mu F^{\nu\mu},
\end{aligned}$$

where the calculation (84) from earlier was used. Now,the divergence of $\Theta^{\mu\nu}$ can either be found by long-hand (extremely tedious) or using the *a-priori* fact that $T^{\mu\nu}$ is conserved.

$$\partial_\mu\Theta^{\mu\nu} = \partial_\mu T^{\mu\nu} - \left(\partial_\mu F^{\lambda\mu}\right)\partial_\lambda A^\nu - F^{\lambda\mu}\partial^2_{\mu\lambda}A_\nu,$$

the first term of which vanishes by Noether's theorem, $\partial_\mu T^{\mu\nu} = 0$, the second vanishes by the field equations and the remaining term $F^{\lambda\mu}\partial^2_{\mu\lambda}A_\nu$ vanishes since $F^{\lambda\mu}$ is skew in $\lambda\,\mu$ but $\partial^2_{\mu\lambda}A_\nu$ is symmetric in $\lambda\,\mu$.

The new object $\Theta^{\mu\nu}$ therefore defines a symmetric, gauge-invariant, trace-free and conserved tensor and is thus a candidate for a physical energy-momentum tensor. Notice also that when the fields are on-shell, the difference $T^{\mu\nu} - \Theta^{\mu\nu}$ is simply a divergence, so that the two tensors describe the same physics.

Exercise 13

The Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2C_\mu C^\mu \ , \text{ where } F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu,$$

has the same dependence on the field derivatives ∂C as the Lagrangian governing electrodynamics in the previous question,

$$\frac{\partial\mathcal{L}}{\partial\,\partial_\mu C_\nu} = F^{\nu\mu}.$$

The field equations are therefore

$$\begin{aligned}
0 &= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu C_\nu}\right) - \frac{\partial\mathcal{L}}{\partial C_\nu} \\
&= \partial_\mu F^{\nu\mu} - m^2C^\nu. \qquad (85)
\end{aligned}$$

Upon taking the divergence we find

$$0 = \partial^2_{\mu\nu}F^{\nu\mu} - m^2\partial_\nu C^\nu.$$

The first term vanishes due to symmetry of second partials and skew symmetry of the field-strength, leaving

$$m^2\partial_\nu C^\nu = 0.$$

If m is non-zero, we see that the fields satisfy

$$\partial_\mu C^\mu = 0.$$

To obtain the field equation for C_0, set $\nu = 0$ in (85),

$$
\begin{aligned}
0 &= \partial_\mu F^{0\mu} - m^2 C^0 \\
\partial_\mu F^{0\mu} &= \partial_i F^{0i} \\
&= \partial_i \left(\partial^0 C^i - \partial^i C^0\right) \\
&= -\partial_i \dot{C}^i - \partial_i \partial^i C^0 \\
\Rightarrow\ 0 &= -\partial_i \dot{C}^i - \partial_i \partial^i C^0 - m^2 C^0 \\
\Rightarrow\ \partial_i \partial^i C_0 + m^2 C_0 &= \partial_i \dot{C}^i,
\end{aligned}
$$

where we variously use the fact that (in the conventions of this course) the Minkowski metric is diag$(-1,1,1,1)$, so that $\partial_0 = -\partial^0$ and $C_0 = -C^0$. The field component C_0 therefore satisfies the inhomogeneous Helmholtz equation

$$
\left(\nabla^2 + m^2\right) C_0 = \partial_i \dot{C}^i,
$$

which, subject to sufficiently nice asympototic behaviour, possesses a unique solution for C_0. For instance, the Green's function for the operator $\nabla^2 + m^2$,

$$
G(\mathbf{x}) = \frac{e^{im|\mathbf{x}|}}{4\pi|\mathbf{x}|},
$$

provides the following solution

$$
C_0(t,\mathbf{x}) = \int_{R^3} d^3y \frac{\partial_i \dot{C}^i(t,\mathbf{y})}{4\pi|\mathbf{y}-\mathbf{x}|} e^{im|\mathbf{y}-\mathbf{x}|}.
$$

Recall that the momenta Π^μ conjugate to the C_μ are defined by

$$
\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{C}_\mu}.
$$

Again using (84), this gives the following expressions

$$
\Pi^\mu = -F^{0\mu} = \begin{cases} 0 & ,\ \mu = 0 \\ -\partial^0 C^i + \partial^i C^0 & ,\ \mu = i \end{cases}
$$

The velocities of the dynamically relevant variables C_i are thus given by

$$
\dot{C}_i = \Pi_i - \partial_i C^0.
$$

Having found this inverse relation between the momenta and the dynamical velocities, the Hamiltonian density can now be computed

$$
\begin{aligned}
\mathcal{H} &= \Pi_\mu \dot{C}^\mu - \mathcal{L} \\
&= \Pi_i \left(\Pi_i - \partial_i C^0\right) + \frac{1}{2} F^{0i} F_{0i} + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} m^2 C_\mu C^\mu
\end{aligned}
$$

The term quadratic in F^{ij} contains no time derivatives and so may be left intact, however, the second term must be re-expressed as a function of the Π^μ,

$$
\begin{aligned}
F^{0i} F_{0i} &= \left(\partial^0 C^i - \partial^i C^0\right)\left(\partial_0 C_i - \partial_i C_0\right) \\
&= \left(-\dot{C}^i - \partial^i C^0\right)\left(\dot{C}_i - \partial_i C_0\right) \\
&= -\Pi^i \Pi_i
\end{aligned}
$$

The complete expression for the Hamiltonian density is therefore

$$
\mathcal{H}[\Pi_\mu, C_\mu] = \frac{1}{2}\Pi_i \Pi^i + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} m^2 C^\mu C_\mu + C^0 \left(\partial_i \Pi^i\right) - \partial_i \left(\Pi^i\, C^0\right)
$$

Where the troubling additional term $-\Pi_i\, \partial^i C_0$ has been written as

$$
C^0 \left(\partial_i \Pi^i\right) - \partial_i \left(\Pi^i\, C^0\right),
$$

which involves an irrelevant 3-divergence term. Since the remainder of the Hamiltonian contains no derivatives in C^0, C^0 may be regarded as a multiplier, that, in the $m = 0$ theory, imposes the constraint $\nabla \cdot \mathbf{\Pi} = m^2 C^0 = 0$, which is precisely Gauss' Law.