Lecture 10

Wednesday, 16 December 2020 09:09

Recap:

- Central Charge in TT OPE: $\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$

  This true for any unitary CFT.

- Transformation law of T under conformal transformation

  $\Rightarrow$ Casimir energy on cylinder $-\frac{1}{12}$.

- Calculated Weyl anomaly $\langle T^\alpha{}_\alpha \rangle = \frac{-c}{12} R_{(2)}$

- Deduced Virasoro algebra from TT OPE & radial quantisation

(iii) Representation of the Virasoro Algebra

Since we have Virasoro symmetry, states will transform in reps of the Virasoro algebra.

What are these reps?

Consider eigenstates of L_0 & $\tilde{L}_0$:

$$L_0 |\psi\rangle = h|\psi\rangle \;\; \& \;\; \tilde{L}_0 |\psi\rangle = \tilde{h} |\psi\rangle$$

These states have energy:

$$\frac{E}{2\pi} = h + \tilde{h} - \frac{c+\tilde{c}}{24}$$

on the cylinder. Let's call $(h, \tilde{h})$ the weights of the state.

Consider

$L_n |\psi\rangle$ & recall $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$

$$L_0 L_n |\psi\rangle = L_n L_0 |\psi\rangle + [L_0, L_n] |\psi\rangle$$
$$= (h-n) L_n |\psi\rangle$$

$L_n|\psi\rangle$ is also an eigenstate of L_0 w/ $(h-n, \tilde{h})$ weight

$\Rightarrow$ Energy is shifted by $-n$.

L_n, $n>0$, lowers weight (energy)

L_0 measures weight (energy)

L_{-n}, $n>0$, raise weight (energy)

If we want spectrum to be bounded from below we must have some states s.t.

$$L_n |\psi\rangle = \tilde{L}_n |\psi\rangle = 0, \quad n>0$$

These states are called primary states

or highest weight states

Representations now come from acting with L_{-n}, $n>0$, on the primary states.

level = eigenvalue of L_0

$$|\psi\rangle \longleftarrow \text{primary state}$$

$$L_{-1}|\psi\rangle$$

$$L_{-1}^2|\psi\rangle \qquad L_{-2}|\psi\rangle$$

$$L_{-1}^3|\psi\rangle \qquad L_{-2}L_{-1}|\psi\rangle \qquad L_{-3}|\psi\rangle$$

$$\vdots$$

Verma module

descendants

& similarly for $\tilde{L}_n$. Focus on L_n now & $\tilde{L}_n$ exactly same.

This infinite tower of states is called a Verma module. These are the irreducible representations of the Virasoro algebra*.

$\Rightarrow$ If we know the spectrum of primary states in CFT, we know the spectrum of all states.

Comments:

- Vacuum $|0\rangle$ is a primary state w/ $h=0$

  $\Rightarrow L_n |0\rangle = 0 \quad \forall \; n \geq 0$

  This is the maximally symmetric state.

  Could we have $L_{-n}|0\rangle = 0$ for $n>0$?

  $$\langle 0| L_n L_{-n} |0\rangle = 2n \langle 0| \cancelto{0}{L_0} |0\rangle + \frac{c}{12} n(n^2-1)\langle 0|0\rangle$$

  $$= \frac{c}{12} n(n^2-1)\langle 0|0\rangle \neq 0 \quad \text{unless } c=0$$

- This is starting to look familiar to OCQ. We will see later that this is related to physical states!

- It could be that some of the states of the Verma module are linearly dependent

  $\Rightarrow$ Verma module might null states.

  This happens for particular values of c & h e.g.

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This happens for particular values of c & h, e.g.

$$c = \frac{2h(5-8h)}{2h+1}$$

w\ primary $|\psi\rangle$ has $L_0|\psi\rangle = h|\psi\rangle$

$$\Rightarrow |\text{null}\rangle = L_{-2}|\psi\rangle - \frac{3}{2(2h+1)} L_{-1}^2 |\psi\rangle$$

* When null states exist, Verma module is reducible.

Null states are also called singular states.

- We have seen primary operators & states w\ weights $(h, \hat{h})$. These are the same thing!

(iv) Unitarity constraints

We want the theory to be unitarity in Minkowski

$\Rightarrow$ Hamiltonian should be Hermitian $\to e^{iHt}$

$$\mathcal{H}_{cyl} = T_{ww} + \bar{T}_{\bar{w}\bar{w}} = \sum_n L_n e^{-in\sigma^+} + \tilde{L}_n e^{-in\sigma^-} - \frac{(c+\tilde{c})}{12}$$

$$\text{Hermitian} \Rightarrow L_n = L_{-n}^{\dagger}$$

What does this imply for our states?

- $h \geq 0$

$$|L_{-1}|\psi\rangle|^2 = \langle\psi|L_1 L_{-1}|\psi\rangle = 2h\langle\psi|\psi\rangle \geq 0$$

NB: only state w\ $h=0$ is the Vacuum.

- $c > 0$:

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$$|L_{-n}|0\rangle|^2 = \frac{c}{12} n(n^2-1)\langle 0|0\rangle \geq 0 \Leftrightarrow c \geq 0$$

If $c=0$, we only have one state $|0\rangle$.

$\Rightarrow$ Non-trivial unitary CFTs have $c>0$.

- Many other similar constraints: can completely classify & solve CFTs w\ $c<1 \rightarrow$ "minimal models"

VI – State-operator map

We will now see that primary states & operators are the same thing.

Normally in QFT local operators live at a point in spacetime. States are defined at fixed τ, ie. on a fixed spatial slice $\Rightarrow$ "very non-local"

E.g. we could write down a "wave-functional"

$\Psi[\phi(\sigma)] \longleftrightarrow$ probability of field having configuration $\phi(\sigma)$ $\forall$ σ @ fixed τ.

However in CFT, we map $\tau \to -\infty$ to a point, the origin of $\mathbb{C}$. So states @ $\tau \to -\infty$ "look" like local operators. We want to think of

$$|\phi\rangle = \underbrace{\lim_{\tau \to -\infty} \phi(\tau, \sigma)|0\rangle}_{\text{non-local}}$$

$$= \underbrace{\lim_{z,\bar{z}\to 0} \phi(z,\bar{z})}_{\text{local in CFT!}} |0\rangle$$

Let's make this more precise:

Recall path integral in QM

$$\langle x_f, \tau_f | x_i, \tau_i \rangle = \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} Dx \, e^{iS}$$

$$= G(x_f, \tau_f ; x_i, \tau_i) \quad \text{propagator}$$

Given wave function $\psi_i(x_i, \tau_i)$ evolves to

$$\psi_f(x_f, \tau_f) = \int dx_i \, G(x_f, \tau_f ; x_i, \tau_i) \psi_i(x_i, \tau_i)$$

$$= \underline{\int dx_i} \int_{x(\tau_i)=\tau_i}^{x(\tau_f)=x_f} Dx \, e^{iS} \underline{\psi_i(x_i, \tau_i)}$$

To find ψ_f & x_f @ τ_f, need to evaluate path integral w/ fixed x_f & τ_f & initial state appears as weight term in the integral over initial conditions

In QFT, we can do the same thing for wave functional

Consider Euclidean cylinder, then initial

$$\psi_i[\phi_i(\sigma), \tau_i]$$

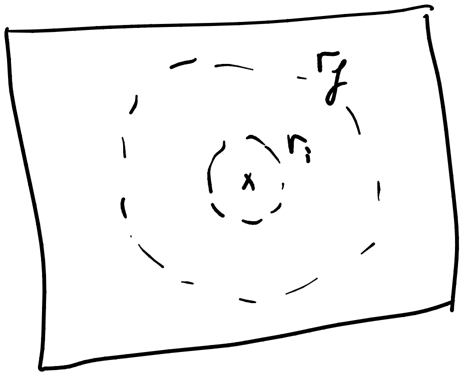
will evolve to ... , Euclidean

will evolve to

$$\Psi_f[\phi_f(\sigma), \tau_f] = \int D\phi_i \int_{\phi(\tau_i)=\phi_i}^{\phi(\tau_f)=\phi_f} D\phi \; e^{-\overset{\text{Euclidean}}{S[\phi]}} \Psi_i[\phi_i(\sigma), \tau_i]$$

Go to $\mathbb{C}$ plane. States live @ fixed radius, so consider $\Psi_i[\phi_i(\sigma), r_i]$ to

$$\Psi_f[\phi_f(\sigma), r_f] = \int D\phi_i \int_{\phi(r_i)=\phi_i}^{\phi(r_f)=\phi_f} \underline{D\phi} \, e^{-S[\phi]} \Psi_i[\phi_i(\sigma), r_i]$$



Now, define initial state at $\tau \to -\infty$, i.e. $r_i = 0$

$$\Psi_f[\phi_f(\sigma), r_f] = \underbrace{\int D\phi_i \int_{\phi(0)=\phi_i}^{\phi(r_f)=\phi_f} D\phi}_{\int^{\phi(r_f)=\phi_f} D\phi} \; e^{-S[\phi]} \, \mathcal{O}(z=0)$$

$$= \int^{\phi(r_f)=\phi_f} D\phi \; e^{-S[\phi]} \, \underline{\mathcal{O}(z=0)}$$

$\Rightarrow$ Any state corresponds to some local operator

$\Rightarrow$ Any state corresponds to some local operator @ $z=0$ & any local operator @ $z=0$ gives me some state.

NB:
- This only works because we can conformally map the cylinder to $\mathbb{C}$. Works in higher dimensions:

$$\mathbb{R}\times S^{D-1} \leftrightarrow \mathbb{R}^D$$

- State-operator map is a 1-to-1 map between states & local operators
- This is not like defining states in Fock space by acting w/ creation operators!

$$|\psi\rangle \sim a^\dagger|0\rangle$$

$a^\dagger$ is not a local operator since it's the Fourier transform of a local operator.

(i) Primary operator & states

What is the vacuum?

$|0\rangle \Leftrightarrow \mathbb{1}$ — identity operator

Sometimes written as $|0\rangle = |\mathbb{1}\rangle$

We can show that primary operators are mapped to primary states

Consider $|\mathcal{O}\rangle = \int \mathcal{D}\phi\, e^{-S[\phi]}\,\mathcal{O}(z=0)$, $\mathcal{O}$ primary consider

$$L_n|\mathcal{O}\rangle = \int D\phi \oint \frac{dz}{2\pi i} z^{n+1} T(z)\, e^{-S[\phi]}\, \mathcal{O}(z=0)$$

$$= \oint \frac{dz}{2\pi i} z^{n+1}\, T(z)\, \mathcal{O}(z=0)$$

$$= \int \frac{dz}{2\pi i} z^{n+1} \left(\frac{h\mathcal{O}}{z^2} + \frac{\partial\mathcal{O}}{z} + \dots \right)$$

- $L_{-1}|\mathcal{O}\rangle = |\partial\mathcal{O}\rangle$ – this is true for all states/operators, even non-primary ones. L_{-1} is the translation operator.

- $L_0|\mathcal{O}\rangle = h|\mathcal{O}\rangle$ – true for any operator of weight h.

- $L_n|\mathcal{O}\rangle = 0 \quad \forall\, n>0$ – true for primary operators only

$$\Updownarrow$$

true for primary states only

$\Rightarrow$ Spectrum of primary operators $\Leftrightarrow$ spectrum of primary states

$\Leftrightarrow$ spectrum of all states on cylinder

(ii) Example: Free scalar field

let's work through comm. relations & state-operator map for free scalar field.

On Euclidean cylinder

On Euclidean cylinder

$$X(w,\bar{w}) = x - i\alpha' p\,\tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\left(\alpha_n e^{inw} + \tilde{\alpha}_n e^{in\bar{w}}\right)$$

w\ reality conditions from Minkowski:

$$\alpha_n^* = \alpha_{-n} \;\&\; \tilde{\alpha}_n^* = \tilde{\alpha}_{-n}$$

Primary field $\partial X = -\sqrt{\frac{\alpha'}{2}} \sum_n \alpha_n e^{inw}$ w\ $\alpha_0 = \sqrt{\frac{\alpha'}{2}}\, p$

weight $1 \Rightarrow$ on $\mathbb{C}$ w\ $z = e^{iw}$:

$$\partial_z X(z) = \left(\frac{\partial z}{\partial w}\right)^{-1} \partial_w X(w) = -i\sqrt{\frac{\alpha'}{2}} \sum_n \frac{\alpha_n}{z^{n+1}}$$

c.f. Exercise Sheet 6, Qu. 1

Inverse relationships:

$$\alpha_n = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dz}{2\pi i}\, z^n\, \partial X(z)$$

$\partial X\, \partial X$ OPE $\Leftrightarrow$ $[\alpha_m, \alpha_n]$:

$$[\alpha_m, \alpha_n] = -\frac{2}{\alpha'}\left(\oint_{|z|>|w|} \frac{dz}{2\pi i} \oint \frac{dw}{2\pi i} - \oint \frac{dw}{2\pi i} \oint_{|z|<|w|} \frac{dz}{2\pi i}\right) \times z^m w^n\, \partial X(z)\, \partial X(w)$$

$$= -\frac{2}{\alpha'} \oint \frac{dw}{2\pi i} \operatorname{Res}_{z=w}\left[z^m w^n\, \partial X(z)\, \partial X(w)\right]$$

$$= -\cancel{\frac{2}{\alpha'}} \oint \frac{dw}{2\pi i} \operatorname{Res}_{z=w}\left[z^m w^n \left(\frac{-\cancel{\alpha'/2}}{(z-w)^2} + \ldots\right)\right]$$

Taylor series around $z=w$

$$= m \oint \frac{dw}{\;}\, w^{m+n-1}$$

series around $z=w$

$$= m \oint \frac{dw}{2\pi i} w^{-m+n-1}$$

$$[\alpha_m, \alpha_n] = m\, \delta_{m+n,0}$$

Now let's look @ state-operator map.

(a) Vacuum: $|0;0\rangle = |\mathbb{1}\rangle \sim \int \mathcal{D}X e^{-S[X]}$

Check this

smooth in path integral

$$\alpha_m |0;0\rangle = \int \mathcal{D}X e^{-S[X]} \oint \frac{dw}{2\pi i} w^m \partial X(w), \quad m \geq 0$$

$$= 0 \quad \forall m \geq 0 \qquad S[X] = \int d^2\sigma\, (\partial X)^2$$

(b) Excited states: $\alpha_{-m}|0;0\rangle = |\partial^m X\rangle, \quad m > 0$

i.e. $\alpha_{-m}|0;0\rangle \sim \int \mathcal{D}X e^{-S[X]} \partial^m X(z=0)$

normalisation

Proof:
$$\alpha_{-m}|0;0\rangle = i\sqrt{\frac{2}{\alpha'}} \int \mathcal{D}X e^{-S[X]} \oint \frac{dw}{2\pi i} w^{-m}\, \partial X(w)$$

$$= \underbrace{i\sqrt{\frac{2}{\alpha'}} \frac{1}{(m-1)!}}_{\text{normalisation factor}} \int \mathcal{D}X e^{-S[X]} \partial^m X(0)$$

(c) Momentum mode: $|0;p\rangle = |e^{ipX}\rangle$

(c) Momentum mode: $|0;p\rangle \cong e^{ipX}$

Symmetry: $X \to X + a$, $e^{ipX} \to e^{ipX} e^{ipa}$

& $|0;p\rangle \to e^{ipa} |0;p\rangle \Rightarrow$ must agree

or compute explicitly:

$$\alpha_n |0;p\rangle = i\sqrt{\frac{2}{\alpha'}} \oint \frac{dw}{2\pi i} w^n \partial X(w) e^{ipX(0)}$$

$$= i\sqrt{\frac{2}{\alpha'}} \oint \frac{dw}{2\pi i} w^n \left(-\frac{i\alpha' p}{2}\right) \frac{e^{ipX}}{w}$$

$$= \sqrt{\frac{\alpha'}{2}}\, p\, \delta_{n,0} |0;p\rangle$$

$$\Rightarrow \alpha_0 |0;p\rangle = \sqrt{\frac{\alpha'}{2}}\, p\, |0;p\rangle, \quad \alpha_{n>0} |0;p\rangle = 0$$
