

Recap:

State-operator correspondence: • 1-to-1 map between local operators & states

- Primary operators $\leftrightarrow$ primary states
- Weights $\leftrightarrow$ weights

Representations of Virasoro algebra: • Constructed from primary states

- Primary $L_n |h\rangle = 0 \quad \forall n > 0$
- $\hookrightarrow$ "highest weight"

Example of state-operator map: • $1 = |0; 0\rangle$

for free scalar field • $e^{ipX} = |0; p\rangle$

- $\partial_m X = \alpha_{-m} |0; p\rangle \quad m > 0$

Non-trivial unitary CFT: • $c > 0$

- $h \geq 0$

reducible { $L_n |h\rangle = 0 \quad \forall n > 0$

Descendant states $L_{-n} |h\rangle$ & repeated action of L 's

$|null\rangle =$ combination of descendant

$\langle null | null \rangle = 0$

irreps: $\frac{\text{Descendant states}}{\text{null states}} = \underline{\text{irrep}}$

⑤ Path Integral Quantisation

I - Faddeev-Popov Ghosts

... leads to a Weyl

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We have seen that $c \neq 0$ leads to a Weyl anomaly:

$$- \langle T^\alpha_\alpha \rangle = - \frac{c}{12} R(\omega)$$

$$- [L_m, L_n] = \underbrace{(m-n)L_{m+n}}_{\text{Witt algebra}} + \underbrace{\frac{c}{12} m(m^2-1) \delta_{m+n,0}}_{\text{central charge} \rightarrow \text{Virasoro algebra}}$$

But $c > 0$ for unitary non-trivial CFT's.

If the Weyl gauge symmetry is anomalous, we cannot recover the classical string, so what can we do?

Let's carefully quantise the string via Path Integral:

$$Z = \frac{1}{\text{Vol}} \int \mathcal{D}g \mathcal{D}X e^{-S_{\text{poly}}[X,g]}$$

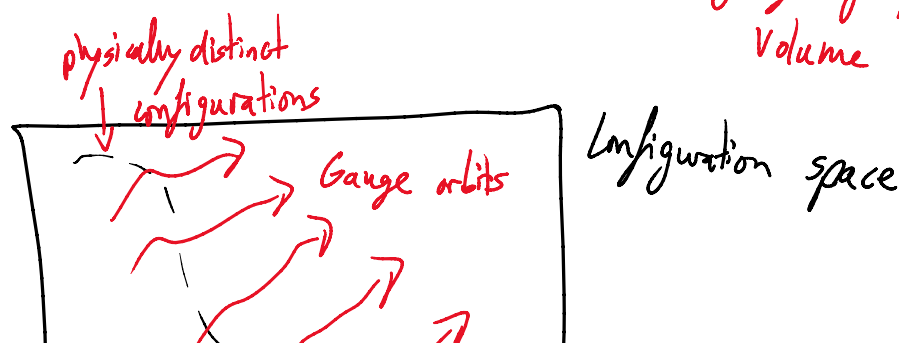
$$S_{\text{poly}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

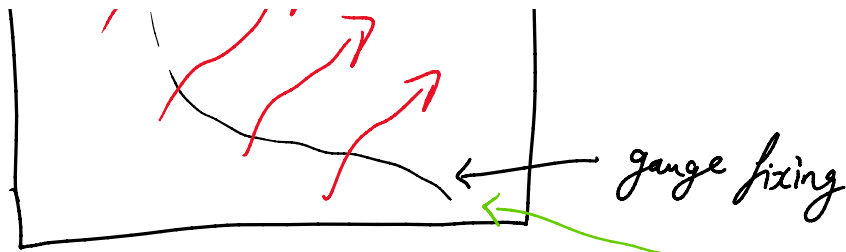
But we have gauge symmetries: (i) Diffeos & (ii) Weyl so that not all field configurations are physically distinct.

We should quotient out by physically equivalent configurations:

$$\frac{1}{\text{Vol}}$$

← gauge group volume





We want to integrate over fields only along
& isolate the integral over gauge orbits.
This gives a Jacobian factor from change of
coordinates.

Let's parameterise the gauge orbits by $\mathcal{Z}$ = diffeos
& Weyl

$\Rightarrow$ Gauge transformation $g \rightarrow g^{\mathcal{Z}}$

w/

$$g_{\alpha\beta}(x) \rightarrow g_{\alpha\beta}^{\mathcal{Z}}(x') = e^{2w(x)} \frac{\partial x^\gamma}{\partial x'^\alpha} \frac{\partial x^\delta}{\partial x'^\beta} g_{\gamma\delta}(x)$$

We already saw that (locally) we can use these to
set

$g \rightarrow$ any fiducial metric $\hat{g}$

So (locally) all g are obtained by gauge transformations
from $\hat{g}$ i.e. for each $g \exists \mathcal{Z}$ s.t. $g = \hat{g}^{\mathcal{Z}}$

$\Rightarrow$ For any functional $F[g]$, we can replace $Dg \rightarrow D\mathcal{Z}$

$$\int Dg F[g] = \int D\hat{g}^{\mathcal{Z}} F[\hat{g}^{\mathcal{Z}}]$$

$$= \int D\mathcal{Z} F[\hat{g}^{\mathcal{Z}}] \left(\det \frac{\delta \hat{g}^{\mathcal{Z}}}{\delta \mathcal{Z}} \right)$$

Jacobian for change
of variables
 $\hat{g}^{\mathcal{Z}} \rightarrow \mathcal{Z}$
- (x)

$$= \int D\zeta \, [Lg] \left(\det \frac{\delta \zeta}{\delta \zeta} \right)^0 \quad (*)$$

$$\det \frac{\delta \hat{g}^3}{\delta \zeta} = \Delta_{FP} [\hat{g}^3] \quad \text{Faddeev-Popov Determinant}$$

There is another way to write Δ_{FP} that is useful:

$$\int D\zeta \, \delta(g - \hat{g}^3) = \Delta_{FP}^{-1} [g] \quad - \quad (**)$$

c.f.

$$\int dx \, \delta(f(x)) = \frac{1}{|f'|} \Big|_{f(x)=0}$$

Two comments

(i) $D\zeta$ is a "Haar measure", i.e. invariant under left/right actions:

$$D\zeta = D(\zeta' \zeta) = D(\zeta \zeta')$$

If this does not hold, we have an anomaly & can't quantise gauge theory

(ii) $\Delta_{FP} [g]$ is gauge invariant: $\Delta_{FP} [g^3] = \Delta_{FP} [g]$

Proof:

$$\begin{aligned} \Delta_{FP}^{-1} [g^3] &= \int D\zeta' \, \delta(g^3 - \hat{g}^{\zeta' \zeta'}) \\ &= \int D\zeta' \, \delta(g - \hat{g}^{\zeta' \zeta'}) \\ &= \int D(\zeta' \zeta') \, \delta(g - \hat{g}^{\zeta' \zeta'}) \end{aligned}$$

invariance of measure

$$= \int \overbrace{D(\tilde{z}^{-1} \tilde{z}')} \delta(g - \hat{g} \tilde{z}^{-1} \tilde{z}') \quad \checkmark \text{ \textit{measure}}$$

$$= \Delta_{FP}^{-1} [g]$$

Going back to path integral:

$$\begin{aligned} Z[\hat{g}] &= \frac{1}{Vol} \int \overbrace{D\tilde{z} Dg DX} \underbrace{\Delta_{FP}[g] \delta(g - \hat{g}^3)}_{\rightarrow = 1 \text{ by } (*)} e^{-S_{poly}[X, g]} \\ &= \frac{1}{Vol} \int D\tilde{z} DX \Delta_{FP}[\hat{g}^3] e^{-S_{poly}[X, \hat{g}^3]} \quad \checkmark \text{ or form } (*) \\ &= \frac{1}{Vol} \int D\tilde{z} DX \Delta_{FP}[g] e^{-S_{poly}[X, \hat{g}]} \quad \downarrow \begin{array}{l} S_{poly} \\ \& \Delta_{FP} \\ \text{are} \\ \text{gauge} \\ \text{invariant} \end{array} \\ &= \int \underline{DX} \underline{\Delta_{FP}[\hat{g}]} e^{-S_{poly}[X, \hat{g}]} \quad \checkmark \text{ nothing depends on } \tilde{z} \end{aligned}$$

let's compute

$$\Delta_{FP}[\hat{g}^3] = \det \frac{\delta \hat{g}^3}{\delta \tilde{z}}$$

But, infinitesimally,

$$\begin{aligned} \delta g_{\alpha\beta} &= \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha + 2\omega g_{\alpha\beta} \\ &= (P \cdot \epsilon)_{\alpha\beta} + 2(\omega - \frac{1}{2} \nabla_\gamma \epsilon^\gamma) g_{\alpha\beta} \end{aligned}$$

$$\delta g_{\alpha\beta} = (P \cdot \epsilon)_{\alpha\beta} + 2\hat{\omega} g_{\alpha\beta}$$

$$\underbrace{\nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha}_{\text{diff/isomorphisms}} - g_{\alpha\beta} \nabla_\gamma \epsilon^\gamma \quad \downarrow \quad \hat{\omega} = \omega + \frac{1}{2} \nabla_\gamma \epsilon^\gamma$$

$P \cdot g = \hat{P}$ $\downarrow$ $\hat{P}$

$$\Rightarrow \det \frac{\delta \hat{g}(\epsilon, \hat{\omega})}{\delta(\epsilon, \hat{\omega})} \Big|_{\hat{g}} = \det \begin{vmatrix} \hat{p} & 0 \\ 0 & 2\hat{g} \end{vmatrix} = \Delta_{FP}[\hat{g}]$$

where $P_{\alpha\beta}^r = 2\delta_{(\alpha}^r D_{\beta)} - g_{\alpha\beta} D^r$
 s.t. $(P \cdot e)_{\alpha\beta} = P_{\alpha\beta}^r e_r \in \mathcal{E}$

How do we compute this?

Recall:

$$\int \prod_i dy_i e^{-y_i A_{ij} y_j} \sim \frac{1}{\sqrt{\det A}}$$

But using Grassmann variables, we can compute $\det A$:

$$\int \prod_i d\psi_i d\theta_i e^{\theta_i A_{ij} \psi_j} \sim \det A$$

where ψ_i & θ_i are Grassmann variables: $\psi_i \theta_j = -\theta_j \psi_i$

$$\psi_i \psi_j = -\psi_j \psi_i$$

$$\theta_i \theta_j = -\theta_j \theta_i$$

Going to continuum limit, we have:

$$\Delta_{FP}[g] = \det \begin{vmatrix} P & 0 \\ 0 & 2g \end{vmatrix}$$

$$= \int D b D b^0 D c D c^0 \exp \left(-\frac{1}{4\pi} \int d^2 x \sqrt{g} (b^{\alpha\beta} P_{\alpha\beta}^r g_r + 2 b^{\alpha\beta} g_{\alpha\beta} c^0) \right)$$

$$= \int \underbrace{D b D b^0 D c D c^0}_{\text{grassmann-valued fields = "Ghosts"}} \exp \left(- \frac{1}{4\pi} \int d^2 \sigma \sqrt{g} (b^{\alpha\beta} P_{\alpha\beta}^{\gamma} c_{\gamma} + 2 b^{0\alpha\beta} g_{\alpha\beta} c^0) \right)$$

convention, absorbed into normalisation of Z

NB: these are spin-0 fields, i.e. have "wrong statistics" for anticommuting fields

We can integrate out $c^0 \Rightarrow b^{0\alpha\beta} g_{\alpha\beta} = 0 \Rightarrow b^{0\alpha\beta}$ is traceless & drops out

NB: $b^{\alpha\beta}$ is symmetric traceless since it multiplies $\underbrace{P_{\alpha\beta}^{\gamma}}_{\substack{\downarrow \\ \text{symmetric traceless}}} c_{\gamma}$

$$\Delta_{FP}[g] = \int D b D c \exp \left(- \frac{1}{2\pi} \int d^2 \sigma \sqrt{g} b^{\alpha\beta} P_{\alpha}^{\gamma} c_{\beta} \right)$$

- S_{ghost}

$$\Rightarrow Z = \int D b D c D X e^{-S_{poly}[X, g] - S_{ghost}[b, c]}$$

$\Rightarrow$ To properly quantise the gauge theory, we introduced auxiliary fields $b, c \rightarrow$ ghosts w/ "wrong" spin-statistics

- b & c appear on same footing as "fundamental" X
- these lead to a non-unitary theory

- these lead to a non-unitary theory
- ghosts remove unphysical gauge degrees of freedom
 $\Rightarrow D-2$ transverse modes of X^M
- physical subsector of theory is unitary
 & decouples from non-unitary part

The definition of the physical subsector occurs via the BRST scheme. *Exercise Sheet 8*

We'll come back to this soon. Let's first study the ghost action.

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2\sigma \sqrt{g} b_{\alpha\beta} \nabla^\alpha c^\beta$$

EOM's: recall $b_{\alpha\beta}$ is symmetric traceless

$$\Rightarrow (P.C)_{\alpha\beta} = \nabla_\alpha c_\beta + \nabla_\beta c_\alpha - g_{\alpha\beta} \nabla_\gamma c^\gamma = 0$$

$$\& \nabla^\alpha b_{\alpha\beta} = 0 \quad (+)$$

Solutions for c EOM are conformal Killing vector fields.

We will understand the geometric meaning of $(+)$ later!

II - Ghost CFT

(i) Classical aspects

In conformal gauge, $\hat{g} = e^{2\omega} \delta_{\alpha\beta} \Rightarrow \sqrt{\hat{g}} = e^{2\omega}$

$$d^2\sigma = \frac{1}{2} d^2z \quad \& \quad \nabla^z = g^{z\bar{z}} \nabla_{\bar{z}} = 2e^{-2\omega} \nabla_{\bar{z}}$$

$$b_\alpha{}^\alpha = 0 \Rightarrow b_{z\bar{z}} = 0$$

$$b_\alpha = 0 \Rightarrow b_{z\bar{z}} = 0$$

$$S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z (b_{zz} \nabla_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \nabla_z c^{\bar{z}})$$

↓ ↓
w factors?

But $\nabla_{\bar{z}} c^z = \partial_{\bar{z}} c^z + \Gamma^z_{\bar{z}\alpha} c^\alpha$

compute $\Gamma^z_{\bar{z}\alpha}$ for $\alpha = z, \bar{z}$

$$\Gamma^z_{\bar{z}\alpha} = \frac{1}{2} g^{z\bar{z}} (\cancel{\partial_{\bar{z}} g_{\bar{z}\alpha}} + \overset{0}{\cancel{\partial_\alpha g_{\bar{z}\bar{z}}}} - \cancel{\partial_{\bar{z}} g_{\alpha\bar{z}}}) = 0$$

$$\Rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z (b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}})$$

Is a CFT! ($\Rightarrow$ Weyl invariant)

This is expected since $\Delta_{\text{FP}}[g]$ is gauge invariant
Weyl invariance w/out scaling $b_{\alpha\beta}$ & c^α !

$\Rightarrow b_{\alpha\beta}$ & c^α are neutral under Weyl transformations

let $b_{zz} = b, \quad b_{\bar{z}\bar{z}} = \bar{b}$
 $c^z = c, \quad c^{\bar{z}} = \bar{c}$

$$\Rightarrow S_{\text{ghost}} = \frac{1}{2\pi} \int d^2z (b \bar{\partial} c + \bar{b} \partial \bar{c})$$

EOM's now simplify: $\bar{\partial} b = \partial \bar{b} = \bar{\partial} c = \partial \bar{c} = 0$

$$\Rightarrow \begin{array}{l} b, c \text{ are holomorphic} \\ \bar{b}, \bar{c} \text{ are anti-holomorphic} \end{array}$$

Energy-momentum tensor Exercise Sheet 9

$$T = 2(\partial c)b + c\partial b$$

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$$\bar{T} = 2(\bar{\partial}\bar{c})\bar{b} + \bar{c}\bar{\partial}\bar{b}$$

(ii) Quantum aspects

We'll focus on holomorphic part w/ obvious extension to anti-holomorphic part.

Propagators:

$$0 = \int Db Dc \frac{\delta}{\delta b(\sigma)} \left[e^{-S_{\text{ghost}}} b(\sigma') \right]$$

$$= \int Db Dc e^{-S_{\text{ghost}}} \left[-\frac{1}{2\pi} \bar{\partial} c(\sigma) b(\sigma') + \delta(\sigma - \sigma') \right]$$

$$\Rightarrow \bar{\partial} c(\sigma) b(\sigma') = 2\pi \delta(\sigma - \sigma')$$

Similarly, $\frac{\delta}{\delta c}$ & c insertion:

$$\bar{\partial} b(\sigma) c(\sigma') = 2\pi \delta(\sigma - \sigma')$$

Using $\bar{\partial} \frac{1}{z} = 2\pi \delta(z, \bar{z})$ to integrate:

$$\boxed{\begin{aligned} b(z) c(w) &= \frac{1}{z-w} \\ c(w) b(z) &= \frac{1}{w-z} \end{aligned}}$$

one implies the other due to Grassmann-valuedness

$$b(z)b(w) = c(z)c(w) = \dots$$

Stress-energy tensors:

$$T(z) = 2:\partial c(z)b(z): + :c(z)b'(z):$$

let's find our primary fields: *Exercise Sheet 7, Qu. 3*

$$T(z) = -\frac{1}{2}:\partial c(z)b(z): + :c(z)b'(z):$$

we find our P_1 / P_2

$$\begin{aligned} T(z)c(w) &= 2: \partial c(z) \underline{b(z)}: \underline{c(w)} + : c(z) \underline{\partial b(z)}: \underline{c(w)} \\ &= 2 \frac{\partial c(z)}{z-w} - \frac{c(z)}{(z-w)^2} + \dots \\ &= -\frac{c(w)}{(z-w)^2} + \frac{\partial c(w)}{(z-w)^2} + \dots \end{aligned}$$

c is primary field of weight -1 c^z

$$\begin{aligned} T(z)b(w) &= 2: \underline{\partial c(z)} \underline{b(z)}: \underline{b(w)} + : c(z) \underline{\partial b(z)}: \underline{b(w)} \\ &= \overset{\text{from anticommuting } c \& b}{\rightarrow} 2 \left(-\frac{1}{(z-w)^2} \right) b(z) - \frac{\partial b(z)}{z-w} + \dots \end{aligned}$$

$$= \frac{2b(w)}{(z-w)^2} + \frac{\partial b(w)}{z-w} + \dots$$

$\Rightarrow b$ is primary of weight $+2$ b_{zz}

The weights of b & c exactly reflect their classical tensorial nature

$$b = b_{zz} \& c = c^z$$

& that they are neutral under Weyl symmetry.

Central charge: *Exercise Sheet 7, qu. 3*

$$T(z)T(w) = \frac{\underline{-13}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

$$\Rightarrow \underline{c = -26}$$