

Recap:

- Path integral quantisation of string
 → Faddeev-Popov procedure

- b & c ghosts: $b_{\alpha\beta}$ & c^α

$\rightarrow b(z), \bar{b}(\bar{z}), c(z), c(\bar{z})$ fields
 weight 2 weight -1

$$\bullet \quad b(z) c(w)' = \frac{1}{z-w}$$

- b & c central charge $C = -26$

- EoM's: $(P.C)_{\alpha\beta} = 0 = \nabla_\alpha C_\beta + \nabla_\beta C_\alpha - g_{\alpha\beta} \nabla_\gamma C^\gamma \leftarrow$ CKV equation
- $\nabla^\alpha b_{\alpha\beta} = 0 \leftarrow$ Moduli of metric

(iii) "Critical Dimension" of String Theory

Gauge fixing diffeo & Weyl symmetry $\Rightarrow$ ghosts with $C = -26$.
Weyl anomaly unless $C = 0$.

$\Rightarrow$ string theory must contain a CFT w/ **critical central charge**

$$C = 26.$$

The simplest possibility is $D=26$ free scalar fields as we have seen so far

$\Rightarrow$ Critical dimension of String theory $D=26$

But we don't need to work w/ free scalar fields.

Any CFT w/ $c=26$ will do e.g.

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$D=4$ free scalar fields

+ CFT_{int} w/ $c=22$

↑
"internal CFT" $\leftrightarrow$ "internal DoF's"

E.g. CFT_{int} may have a geometric interpretation, e.g.

a compactification, or it may not, e.g.

"non-geometric backgrounds", "asymmetric orbifolds", ...

Superstrings: have • local supersymmetry
 $\Rightarrow$ extra ghosts " β - γ systems"
w/ $c=11$

• worldsheet fermions 1 fermion for each 1 boson
 $\Rightarrow c = \frac{1}{2}$ cf. Exercise Sheets 6, eq. 6

$$\Rightarrow \text{Must have } D(1 + \underbrace{\frac{1}{2}}_{\substack{\text{w} \\ \text{X}}}) = 26 - \underbrace{11}_{\substack{\text{w} \\ \text{bc}}} = 15$$

$$\Rightarrow \underline{D=10.}$$

(iv) Ghost number symmetry

$$S_{gh} = \frac{1}{2\pi} \int d^2z (b \bar{\partial} c + \bar{b} \partial c)$$

The ghost action is invariant under $U(1)_{gh}$ global symmetry:

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$$\delta b = -i \epsilon b$$

$$\delta c = i \epsilon c$$

& similarly for $\bar{b}, \bar{c}$

$\Rightarrow$ Ghost number symmetry

Ghost number current

$$j_{gh} = - : b(z) c(z) : \quad \& \text{ similarly for } \bar{j}_{gh}(\bar{z})$$

w/ conserved charge

$$N_{gh} = \frac{1}{2\pi i} \oint dz \, j_{gh}(z)$$

check using $b(w) c(z) = \frac{1}{w-z} + \dots$ that

$$[N_{gh}, c(z)] = c(z)$$

$$[N_{gh}, b(z)] = -b(z)$$

$$\begin{aligned} [N_{gh}, c(z)] &= - \frac{1}{2\pi i} \oint_z dw : b(w) c(w) : c(z) \\ &= \frac{1}{2\pi i} \oint_z dw \left(c(w) \frac{1}{w-z} + \dots \right) \end{aligned}$$

anticommute $\Rightarrow (-1)$

$$= c(z)$$

$$[N_{gh}, b(z)] = -b(z) \quad \text{w/ - sign due to Grassmann nature of } b \& c$$

III - BRST quantisation

let us revisit the spectrum covariantly. After gauge fixing, our theory consists X, b, c fields
& we know it's not unitary since $c=0$.
How do we find the physical spectrum?

(i) BRST symmetry

The gauge-fixed action

$$S = S_{\text{poly}}[X; \hat{g}] + S_{\text{gh}}[b, c; \hat{g}]$$

has a new global fermionic symmetry: (up to EOM's)

$$\delta_{\theta} X^M = i \theta \overset{\text{Grassmann valued}}{c^{\alpha}} \partial_{\alpha} X^M$$

c.f. Exercise Sheets
8 & 9

$$\delta_{\theta} c^{\alpha} = i \theta c^{\beta} \partial_{\beta} c^{\alpha}$$

$$\delta_{\theta} b_{\alpha\beta} = -i \theta (T_{\alpha\beta}^{(X)} + T_{\alpha\beta}^{(\text{gh})})$$

This is known as BRST symmetry (Becchi-Rouet-Stora-Tyutin)
it is nilpotent:

$$\boxed{\delta_{\theta} \delta_{\theta'} = 0}$$

(up to EOM's)

$\Rightarrow$ the conserved charge Q_B s.t.

$$[Q_B, \dots] = 0$$

$$\Theta \overset{\text{fermionic}}{\uparrow} Q_B = \int \Theta$$

is a nilpotent operator

$$Q_B^2 = \frac{1}{2} \{Q_B, Q_B\} = 0$$

The physical states are counted by BRST-cohomology:

States s.t. $Q_B |\psi\rangle = 0$ are called Q_B -closed
 $\in \text{Ker } Q_B$

$\chi |\psi\rangle = Q_B |\chi\rangle$ are called Q_B -exact
 $\in \text{Im } Q_B$

$Q_B |\psi\rangle = Q_B^2 |\chi\rangle = 0$ is also closed

BRST cohomology $\mathcal{H}_{\text{BRST}}$ is defined as

$$\mathcal{H}_{\text{BRST}} = \frac{\text{Ker } Q_B}{\text{Im } Q_B} = \frac{\mathcal{H}_{\text{closed}}}{\mathcal{H}_{\text{exact}}}$$

Physical states satisfy

$$Q_B |\text{phys}\rangle = 0$$

... .. $Q_B |\text{ghost}\rangle \neq 0$

$$Q_B |\text{phys}\rangle = 0$$

but $|\text{phys}\rangle \sim |\text{phys}\rangle + Q_B |\text{arbitrary}\rangle \leftarrow \text{realises gauge symmetry}$
are identified

$$\left(\begin{array}{l} \text{E.g. de Rham cohomology: } dW_{(p)} = \Omega_{(p+1)} \\ d^2 = 0 \end{array} \right)$$

To find physical states we must impose another condition:

The physical spectrum is given by BRST cohomology of operators of the form $C(z)V^{(m)}$

Polchinski 4.4

eg. Polakov
this would be operators constructed from X

(ii) BRST anomaly

So far, the discussion has been essentially classical.
Let's briefly look at quantum theory.

Associated to BRST symmetry is the current:

$$j_B = c \overset{\uparrow}{T^{(m)}} + \frac{1}{2} : c \overset{\uparrow}{T^{(gh)}} : + \underbrace{\frac{3}{2} \partial^2 c}_{\text{total derivative}}$$

T for a general
matter CFT, e.g. $T(w)$
for Polyakov

T of ghost

total derivative
 $\Rightarrow$ does not change
 Q_B but makes
 j_B primary!

NB: Q_B has ghost number +1

We can compute

$$\{Q_B, Q_B\} = \oint \frac{dw}{2\pi i} \oint \frac{dz}{2\pi i} j_B(w) j_B(z)$$

contour centered
@ w

$$= \oint \frac{dw}{2\pi i} \operatorname{Res}_{z \rightarrow w} (j_B(w) j_B(z))$$

$$\stackrel{!}{=} 0 \text{ for BRST quantisation to work!}$$

$$\text{But } j_B(w) j_B(z) = \dots - \frac{c^m - 26}{z - w} : c(w) \partial^3 c(w) : + \dots$$

higher order singularities
Problematic

BRST anomaly & BRST quantisation fails unless

$$\underline{c^m = 26} \Leftrightarrow \text{Critical central } 26!$$

(iii) Physical spectrum

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(i) $C \times V^{(m)}$

We can write physical operators as

$$\Theta(z, \bar{z}) = : \underset{\substack{\uparrow \\ \text{anti-hol.} \\ \text{part}}}{C(z)} \underset{\substack{\uparrow \\ \text{matter operator}}}{\bar{C}(\bar{z})} V^{(m)}(z, \bar{z}) :$$

(ii) Q_B closed

$$Q_B \Theta(z, \bar{z}) = \text{Res}_{z \rightarrow w} (j_B(w) \Theta(z, \bar{z})) \quad \& \quad \text{similarly for } Q_{\bar{B}}$$

focus on holomorphic part

$$= \text{Res}_{z \rightarrow w} \left(C(w) T^{(m)}(w) \Theta(z) + \frac{1}{2} : C(w) T^{(gh)}(w) \Theta(z) : \right)$$

The ghost & matter parts decouple:

$$Q_B C(z) = \text{Res}_{w \rightarrow z} \left(\frac{1}{2} : C(w) T^{(gh)}(w) : C(z) \right)$$

$$= \text{Res}_{w \rightarrow z} \left(\frac{C(w) \partial C(w)}{w - z} + \dots \right)$$

$$= C(z) \partial C(z)$$

matches $\delta_\Theta C^\alpha = C^\beta \partial_\beta C^\alpha$

$$Q_B V^{(m)}(z) = \text{Res}_{w \rightarrow z} \left(: \underbrace{C(w)}_{\text{Taylor series}} \underbrace{T^{(m)}(w)}_{\text{or } B} V^{(m)}(z) : \right)$$

$$= \text{Res}_{w \rightarrow z} \left[: C(z) T^{(m)}(w) V^{(m)}(z) : \right] \quad \leftarrow \text{we pick up 1st order pole TV OPE}$$

$$= w \rightarrow z \left[c(z) + \sum_{n=1}^{\infty} \frac{1}{n!} : \partial^n c(z) T^{(m)}(w) V^{(m)}(z) : (w-z)^n \right]$$

pole TV OPE

we pick up $(n+1)$ -th order pole of TV OPE

$$= c(z) \partial V^{(m)}(z) + \sum_{n=1}^{\infty} \frac{1}{n!} : \partial^n c(z) [T^{(m)} V^{(m)}]^{(n+1)}(z) :$$

$(n+1)$ -th order pole of $T-V$ OPE

$$Q_B(c(z)V^{(m)}(z)) = (Q_B c(z))V^{(m)}(z) - c(z)Q_B V^{(m)}(z) = 0$$

$$\Rightarrow 0 \doteq c(z) \partial c(z) V^{(m)}(z) - c(z) \cancel{\partial c(z)} \partial V^{(m)}(z) - \sum_{n=1}^{\infty} \frac{1}{n!} : \partial^n c(z) [T^{(m)} V^{(m)}]^{(n+1)}(z) :$$

$$\Rightarrow \text{We require } [T^{(m)} V^{(m)}]^{(n+1)}(z) = \begin{cases} V^{(m)}(z), & n=1 \\ 0, & n>1 \end{cases}$$

$$\Rightarrow T^{(m)}(w) V^{(m)}(z) = \frac{V^{(m)}(z)}{(w-z)^2} + \frac{\partial V^{(m)}(z)}{w-z} + \dots$$

$\Rightarrow V^{(m)}(z, \bar{z})$ must be a primary operator of weight $(h, \bar{h}) = (1, 1)$



$\dots, V^{(m)}, \dots$

$$\Downarrow \\ Q_B(c(z)\bar{c}(\bar{z})V^{(m)}(\bar{z},\bar{z})) = 0 \quad \& \quad \bar{Q}_B(1) = 0$$

This is not so surprising:

$Q_B \sim$ conformal transformation (C-valued)

Requiring $cV(\bar{z})$ to be Q_B closed

$\Rightarrow cV(\bar{z})$ to be conformally invariant

$\downarrow$ weight = -1 $\rightarrow$ weight +1

(iv) Example: $D=26$ free scalar fields

Level 0: only primary operator is $:e^{ip \cdot X}:_0$

$$\text{weight } (h, \tilde{h}) = \left(\frac{\alpha' p^2}{4}, \frac{\alpha' p^2}{4} \right)$$

$$\Rightarrow \text{Physical fields: } \frac{\alpha' p^2}{4} = 1 \Rightarrow M^2 = -p^2 = -\frac{4}{\alpha'}$$

Mass-shell condition for tachyon from LQ

Level 1: Primary operators : $\overset{\text{constant}}{\downarrow} \sum_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ip \cdot X}_0$

$$\text{primary} \Leftrightarrow \underline{\sum_{\mu\nu} p^\nu = \sum_{\mu\nu} p^\mu = 0}$$

c.f. Exercise Sheet 6, Qu. 5

$$\text{Weight } (h, \tilde{h}) = \left(1 + \frac{\alpha' p^2}{4}, 1 + \frac{\alpha' p^2}{4} \right)$$

$$\text{Weight } (h, \bar{h}) = \left(1 + \frac{\alpha' p^-}{4}, 1 + \frac{\alpha' p^2}{4}\right)$$

$$= (1, 1) \Rightarrow \boxed{p^2 = 0}$$

Mass-shell condition for level 1
from LCC

Covariant way of obtaining spectrum as seen in LCC.

Q_B -exactness

Consider level 1: $\sigma(\tau, \bar{\tau}) = :c\bar{c} \zeta_{\mu\nu} \partial X^\mu \bar{\partial} X^\nu e^{ip \cdot X}: (\tau, \bar{\tau})$ ghost number 1

Q_B has ghost number 1

Consider ghost number 0

$$\lambda \sim : \bar{c} \zeta_\nu \bar{\partial} X^\nu e^{ip \cdot X}: (\tau, \bar{\tau}) \quad \text{w/ } p \cdot \zeta = 0$$

$$Q_B \lambda \sim : c\bar{c} p_\mu \zeta_\nu \partial X^\mu \bar{\partial} X^\nu e^{ip \cdot X}:$$

$$\begin{aligned} \Rightarrow V \rightarrow V + Q_B \lambda &\Leftrightarrow \zeta_{\mu\nu} \rightarrow \zeta_{\mu\nu} + p_\mu \zeta_\nu \\ \bar{Q}_B \lambda &\Leftrightarrow \zeta_{\mu\nu} \rightarrow \zeta_{\mu\nu} + \zeta'_\mu p_\nu \end{aligned} \quad \left. \vphantom{\begin{aligned} \Rightarrow V \rightarrow V + Q_B \lambda \\ \bar{Q}_B \lambda \end{aligned}} \right\} \begin{array}{l} \text{gauge} \\ \text{symmetries} \\ \text{of level 1} \\ \text{massless} \\ \text{fields} \end{array}$$