Recap:

- Critical central charge $\to$ $c=26$
  $\to$ different possibilities
- Physical states $\in$ BRST cohomology
  & $\mathcal{O} = c\bar{c}V^{(m)}$
- Q_B closed $\Rightarrow$ $V^{(m)}$ = primary of weight $(1,1)$
  $\Rightarrow$ Mass-shell condition for physical states
- $Q_B^2 = 0 \Longleftrightarrow c = 26$

$\boxed{\text{$Q_B, \bar{Q}_B$ - closed states that are physical are of the form } \mathcal{O}(z,\bar{z}) = :c\bar{c}V^{(m)}:(z,\bar{z}) \text{ w/ } V^{(m)} \text{ is a primary operator of weight } (h,\tilde{h})=(1,1)}$

NB: given a primary of weight $(h,\tilde{h})=(1,1)$, we can create a BRST-closed operator:

$$\mathcal{V} = \underbrace{\int d^2z}_{\text{weight }(-1,-1)}\, \underbrace{V^{(m)}(z,\bar{z})}_{\text{weight }(1,1)}$$

This follows since

$$[Q_B, V^{(m)}(z,\bar{z})] = \partial\left(c(z)V^{(m)}(z,\bar{z})\right)$$

is a total derivative

$$\Rightarrow [Q_B, \mathcal{V}] = 0$$

Physical states $\Longleftrightarrow$ Primary operators of weight $(1,1)$

For each such state, we can create 2 BRST-closed operators:

- Unintegrated Vertex Operator

operators:

- Unintegrated Vertex operator
$$\boxed{U(z,\bar{z}) = :c(z)\,\bar{c}(\bar{z})\,V^{(m)}(z,\bar{z}):}$$
- Integrated vertex operator
$$\mathcal{V} = \int d^2z\, V^{(m)}(z,\bar{z})$$

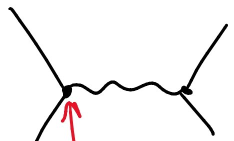
We will use these to compute scattering amplitudes.

⑥ String Interactions

(i) Sum over topologies

How do strings interact?

For point particles, we would look @ scattering processes like:



In QFT, we have to choose interaction terms but in string theory we have no choice.

When first discussing Polyakov action,

$$S_{Poly} = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

we argued this was the only weyl-invariant action.

Actually, we could also add a term:

2-d Ricci scalar

$$S_\chi = \lambda\chi = \lambda \frac{1}{4\pi}\int d^2\sigma \sqrt{g}\, R_{(2)}$$

But $\sqrt{g}R_{(2)}$ is a total derivative & so it does not contribute to dynamics

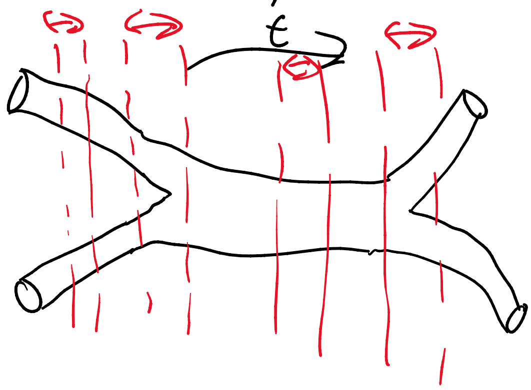
But $\sqrt{g}R(\sigma)$ is a total ... contribute to dynamics

Actually, χ = "Euler number" is a topological invariant that is important interactions as we'll see.

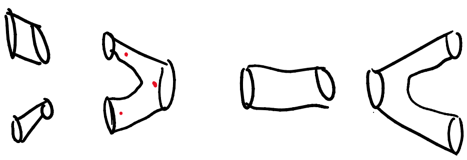
$\Rightarrow$ In Minkowski space, string theory is free on worldsheet.

So how do strings interact?

Let's draw a "Feynman diagram"

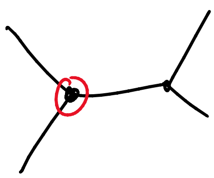


At any given time interval, we have a worldsheet that looks like cylinder we've studied so far

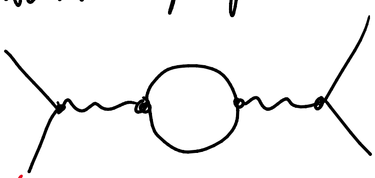


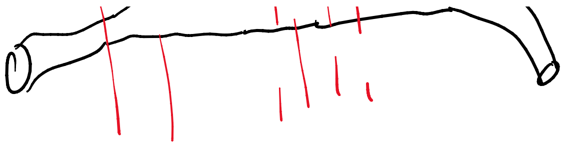
Locally, we do not "see" interactions.

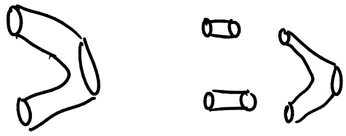
c.f. QFT



What would loop processes look like?







Again, locally same description → what we've studied so far. Loop processes correspond to globally different worldsheets:

loops come from extra holes in the worldsheet.

$\Rightarrow$ Tree, 1-loop, 2-loop, etc. correspond to topologically distinct worldsheets

We see that string theory encodes interactions via different topologies of its worldsheet & the only choice we have is what topologies to include, (ignoring endsource boundaries)

(i) no boundaries $\Leftrightarrow$ closed string amplitudes

(ii) including boundaries $\Leftrightarrow$ open string amplitudes

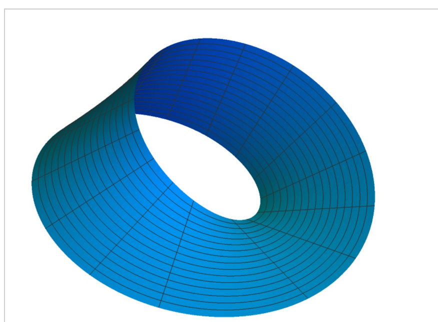
(iii) oriented w/ out boundaries $\Leftrightarrow$ oriented, closed string — we will focus on this

(iv) oriented incl. boundaries $\Leftrightarrow$ oriented, open string

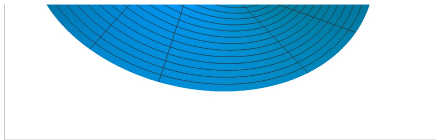
Examples of unoriented worldsheets:

(i) Möbius strip — See also Moodle notebook → & www.emanuelmalek.com/teaching/StringTheory/RiemannSurfaces.html

(ii) Klein bottle



(ii) Klein bottle Riemann surfaces.html



The path integral of string theory includes a sum over topologies:

$$Z = \sum_{\text{topologies}} Z^{(\text{topologies})}$$

w\

fiducial metric on topology

$$Z^{(\text{topology})} \sim \int DX\, Db\, Dc\; e^{-S_{Poly}[X;\hat{g}] - S_{ghost}[b,c;\hat{g}] - \lambda \chi}$$

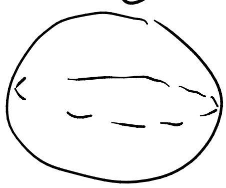
this now gives a weighting to different topologies

Every compact* connected 2-dim manifold is topologically equivalent to a sphere w\ g handles, b holes (boundaries) & c cross-caps

*We will see soon why compact

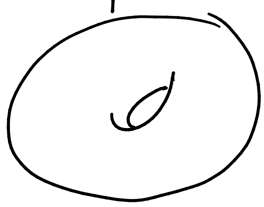
E.g. $\boxed{b = c = 0}$

S^2



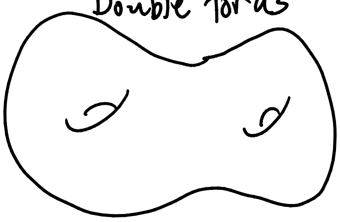
$g = 0$

T^2



$g = 1$

Double torus



$g = 2$

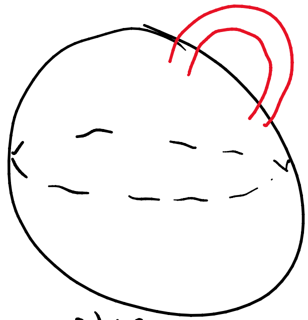
g = Genus of surface

$$\boxed{\text{Euler number } \chi = 2 - 2g \overbrace{- b - c}^{\text{for open/unoriented strings}}}$$

But each handle corresponds to a string loop:

tree-level

$\underline{e^{-\lambda\chi} = e^{-2\lambda}}$



$\underline{e^{-2\lambda+2\lambda}}$

$e^{\lambda} \equiv g_s$ plays the role of string coupling constant

Topologies are weighted w/ $g_s^{-\chi}$

When $g_s \ll 1$, we can use perturbation theory over topologies.

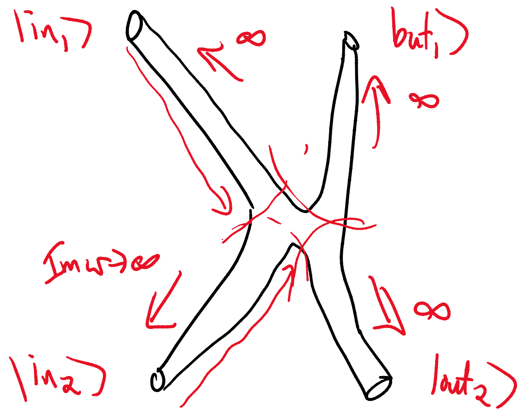
NB: we will see that g_s is not a free parameter!

(ii) Vertex Operators

How do we think of in/out states?

Technically, it is very hard to compute correlation functions.

$\rightarrow$ We will compute S-matrix amplitudes w/ asymptotic <u>on-shell</u> states @ ∞



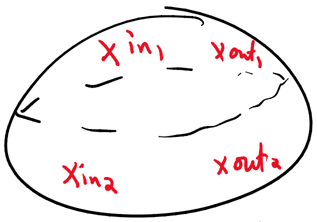
$|in_2\rangle$ w/ ... to $|out_2\rangle$

We can conformally transform the ∞-ly long cylinders w\

$$z = e^{-iw}$$

to a disc w\ $|in_2\rangle \to$ origin of the disc

We get a sphere:



w\ 4 insertions corresponding to $|in_1\rangle, |in_2\rangle, |out_1\rangle, |out_2\rangle$

This is exactly the state-operator map!

I.e. for each asymptotic state, we insert an operator into compact worldsheet correspond to that state!

The S-matrix should be gauge invariant

$\Rightarrow$ we need to use vertex operators corresponding to the states we are scattering:

$$\int d^2z\, V^{(m)}(z,\bar{z})$$

or

$$c(z)\,\bar{c}(\bar{z})\, V^{(m)}(z,\bar{z})$$

Since the result should not depend on the choice of insertion positions, we will typically use integrated vertex operators.

We see that for n-point scattering between states Λ_i, spacetime momenta p_i, we compute:

vertex operators

states Λ_i, spacetime momenta p_i, we compute:

vertex operators

$$A^{(n)}(\Lambda_i; p_i) = \sum_{\text{topologies}} g_s^{-\chi} \int DX\, Db\, Dc\, e^{-S_{\text{poly}} - S_{\text{ghost}}} \prod_{i=1}^{n} V_{\Lambda_i}(p_i)$$

(iii) Global issues

We saw that string perturbation series is run over topologies

$\Rightarrow$ Need to understand global issues that we've ignored.

When computing Δ_{FP}, we fixed $g \to \hat{g}$ and

(a) ignored the remnant conformal symmetry which leaves $\hat{g}$ invariant $\Rightarrow$ We still have remnant gauge symmetries

$\rightarrow$ Conformal Killing Vectors (CKV) which are globally well-defined

(b) assumed that any g can be gauge transformed into $\hat{g}$ $\Rightarrow$ Not all g are conformally equivalent!

$\rightarrow$ Metric moduli

CKV's we already discussed:

$$(P \cdot v)_{\alpha\beta} = 0$$

$$(P \cdot v)_{\alpha\beta} \equiv \nabla_\alpha v_\beta + \nabla_\beta v_\alpha - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma v^\gamma$$

In flat gauge, these are holomorphic vector fields

$\Rightarrow$ locally ∞-many of these

Globally, these are not all well-defined

The globally well-defined CKV's generate conformal Killing Group (CKG) & we still need to gauge fix this!

fix this!