

Recap:

- Interactions do not require modification of action
- Scattering amplitudes $\rightarrow$ loop diagrams correspond to diff. topologies
 - $\rightarrow$ Path integral includes sum over topologies
 - $\rightarrow$ $e^{-\chi}$ weights different topologies
 - $\rightarrow g \sim e^{-\lambda}$
- In/Out states via state operator map become operator insertions



- Global properties are important!
 - $\rightarrow$ Not every metric is related by gauge transformations
 - $\rightarrow$ Remnant gauge symmetries
- Orientable closed 2-manifolds are $\sim$ spheres w/ g handles
 - $\hookrightarrow$ genus $\chi = 2 - 2g$
- $\chi = 2 \Leftrightarrow S^2 \leftarrow$ tree-level amplitude
- $\chi = 0 \Leftrightarrow T^2 \leftarrow$ 1-loop amplitude

last time! (i) CKV's:

$$(P \cdot v)_{\alpha\beta} = D_{\alpha} v_{\beta} - D_{\beta} v_{\alpha} - g_{\alpha\beta} D_{\gamma} v^{\gamma} = 0$$

$\hookrightarrow$ holomorphic vectors in flat gauge

↳ holomorphic vectors in flat gauge
 Globally well-defined Killing vector fields
 generate CKG $\rightarrow$ remnant symmetry group

Metric moduli

Consider a variation of the metric

$$g_{\alpha\beta} \rightarrow g_{\alpha\beta} + \delta g_{\alpha\beta}$$

When can this be undone by a diffeo + Weyl?

Can use Weyl to make $\delta g_{\alpha\beta}$ traceless.

Now consider the +ve definite inner product
 (Euclidean worksheet!!) defined as

$$(t_{(2)}, t'_{(2)}) \equiv \int d^3x \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} t_{\alpha\beta} t'_{\gamma\delta}$$

$$(v, v') \equiv \int d^3x \sqrt{g} g_{\alpha\beta} v^\alpha v'^\beta$$

& similarly for other tensors

We can define $P^\dagger$ wrt $(,)$:

$$(P^\dagger t_{(2)}, v) = (t_{(2)}, \underbrace{P v}) \quad \forall t_{(2)}, v$$

$$P: \Gamma(M) \rightarrow \Gamma(T^*M \otimes_{S_t} T^*M)$$

$\underbrace{\quad}_{\text{symm. traceless}}$

NB: $P^\dagger$ sends symm. traceless 2-tensors to vectors.

NB: P' sends symm. traceless 2-tensors to vectors.

Claim: Metric moduli are symm. traceless 2-tensors
s.t. $P^\dagger \delta g = 0$

Proof: Assume $\exists$ some diffeo s.t.

$$\nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \epsilon^\gamma = \delta g_{\alpha\beta}$$

Then we would need to have

$$(\underbrace{\delta g, P. \epsilon}_{\delta g}) \neq 0$$

But $(\delta g, P. \epsilon) = (P^\dagger \delta g, \epsilon) = 0$

$\Rightarrow$ ~~is~~ a diffeo/Weyl transformation that undoes δg !

What is $P^\dagger$?

$$(t_{(2)}, P. \epsilon) = \int d^2\sigma \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} t_{\alpha\beta} (P. \epsilon)_{\gamma\delta}$$

$$= 2 \int d^2\sigma \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} t_{\alpha\beta} \nabla_\gamma \epsilon_\delta$$

$$= -2 \int d^2\sigma \sqrt{g} g^{\alpha\gamma} g^{\beta\delta} (\nabla_\gamma t_{\alpha\beta}) \epsilon_\delta$$

$$= -2 \int d^2\sigma \sqrt{g} (\nabla^\beta t_{\alpha\beta}) \epsilon^\alpha$$

$$(\nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha - g_{\alpha\beta} \nabla_\gamma \epsilon^\gamma)$$

$$= -2 \int d^4x \sqrt{g} (\nabla^\beta t_{\alpha\beta}) \in \mathcal{A}$$

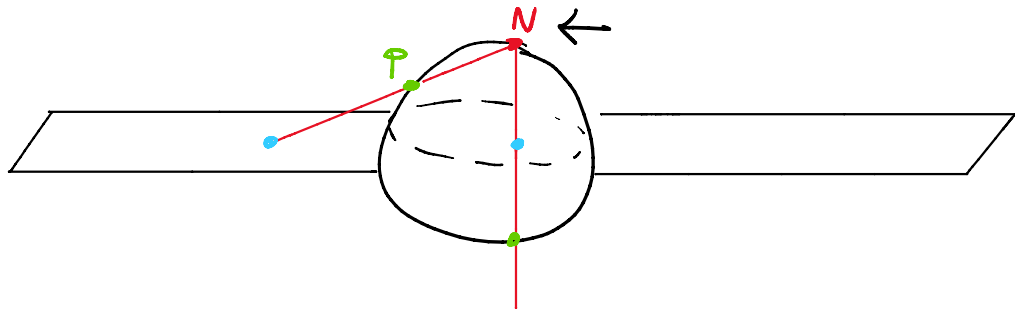
$$= (p^\dagger t_{(2)}, \in)$$

$$(p^\dagger t_{(2)})_\alpha = -2 \nabla^\beta t_{\alpha\beta} \rightarrow \text{recall this is the } \mathcal{L} \text{ EOM}$$

NB: Symmetric 2-tensors are often called "quadratic differentials".

(iv) Example: The Sphere

$S^2 \simeq \mathbb{C} \cup \{\infty\}$ c.f. stereographic projection:



Each point, P on S^2 gets mapped to point on equatorial plane that intersects straight line through P & N .
 All points $P \neq N$ get mapped to plane $(\mathbb{C})$ (1-to-1)
 N gets mapped to $\infty \Rightarrow S^2 \simeq \mathbb{C} \cup \infty$.

NB: Pulling back the flat metric on $\mathbb{C}$ to S^2 gives the conformally flat metric

$$ds^2 = \frac{4R^2}{(1+|z|^2)^2} dz d\bar{z}$$

$$ds = \frac{1}{(1+|z|^2)^2} dz d\bar{z}$$

$\Rightarrow$ We can use flat gauge on S^2

$$\text{CKV: } p. \epsilon = 0 \rightarrow \bar{\partial} \epsilon = \partial \bar{\epsilon} = 0$$

$$\begin{array}{l} \text{Metric: } p. t = 0 \rightarrow \bar{\partial} t = \partial \bar{t} = 0 \\ \text{Moduli} \end{array} \left. \begin{array}{l} t_{z\bar{z}} = t \\ t_{\bar{z}z} = \bar{t} \end{array} \right\} \begin{array}{l} \text{c.f. b} \\ \text{ghost} \\ \text{EoM} \end{array}$$

So we are looking for well-defined holomorphic

vector fields $\Leftrightarrow$ CKV

symm. traceless
2-tensors $\Leftrightarrow$ moduli

(a) Conformal Killing Group

We need to look for hol. vector fields on $\mathbb{C}$ that are also well-defined @ $z = \infty$

Change coordinates $z = \frac{1}{u}$

$$\epsilon \equiv \epsilon^z \rightarrow \epsilon^u = \frac{\partial u}{\partial z} \epsilon^z = -\frac{1}{z^2} \epsilon^z$$

So we need holomorphic ϵ on $\mathbb{C}$ that do not grow faster than z^2 as $z \rightarrow \infty$

$$\Rightarrow \epsilon = \alpha + Bz + \gamma z^2$$

$$\Rightarrow \underline{\epsilon = \alpha + \beta z + \gamma z^2}$$

are globally well-defined CKV's

$\Rightarrow$ 3 independent hol. & 3 anti-hol. CKV's

c.f. Exercise Sheet 10, Qu. 2

These form the global Conformal Killing Group

$$\underline{SL(2, \mathbb{C}) / \mathbb{Z}_2 \sim PSL(2, \mathbb{C})}$$

This acts as

$$z \rightarrow \frac{az+b}{cz+d}$$

Fraction Linear Transformation

& wlog consider $ad-bc=1$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$$

But also $(a, b, c, d) \rightarrow (-a, -b, -c, -d)$ gives same transformation

$\Rightarrow$ Quotient by $\mathbb{Z}_2$ to get $PSL(2, \mathbb{C}) \sim \frac{SL(2, \mathbb{C})}{\mathbb{Z}_2}$.

This is also often called the Möbius Group.

When we consider tree-level scattering amplitudes of closed strings (i.e. S^2 worldsheet)

we will need to fix remnant gauge symmetry

we will need to fix remnant gauge symmetry $PSL(2, \mathbb{C})$. We will do this by fixing the position of three vertex operators.

(b) Moduli

Need well-defined hol. symmetric 2-tensors on $\mathbb{C} \setminus \{\infty\}$
 Define $u = \frac{1}{z}$, what are the well-defined hol. sym. 2-tensors?

$$t_{zz} \equiv t \rightarrow t_{uu} = \left(\frac{\partial z}{\partial u}\right)^2 t_{zz} = u^{-4} t_{zz} = z^4 t_{zz}$$

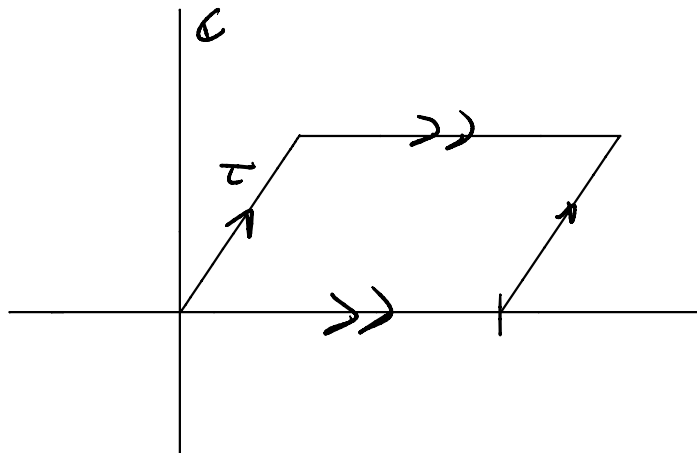
is non-singular as $z \rightarrow \infty$ if t_{zz} grows slower than z^{-4} as $z \rightarrow \infty$

$\Rightarrow$ there are no moduli on S^2

(v) Example: The Torus

Define the torus as $\mathbb{C}/\Lambda$ lattice s.t.

$$\begin{aligned} z &\sim z+1 \\ z &\sim z+\tau \\ \text{Im } \tau &> 0 \end{aligned}$$



We can choose the metric on T^2



$\Rightarrow 2$ moduli

How can we see these moduli directly?

We started w/ $\mathbb{C}/\Lambda$ s.t. $z \sim z+1, z \sim z+\tau, \text{Im } \tau > 0$

$$\& \ ds^2 = dz d\bar{z}$$

Equivalently, we could change to real coords

$$\left. \begin{aligned} \sigma^1 &= \frac{1}{2}(z + \bar{z}) \\ \sigma^2 &= \frac{1}{2}\left(\frac{z}{\tau} + \frac{\bar{z}}{\bar{\tau}}\right) \end{aligned} \right\} \begin{aligned} \sigma^1 &\sim \sigma^1 + 1 \\ \sigma^2 &\sim \sigma^2 + 1 \end{aligned} \quad \left| \begin{array}{l} \text{standard} \\ \text{periodicity} \end{array} \right.$$

$$\Rightarrow \text{metric } ds^2 = \underbrace{|d\sigma^1 + \tau d\sigma^2|}_{\text{metric picks up } \tau!}$$

c.f. Exercise sheet 10, Eq. 1

We can (i) keep periodicity fixed & consider different metrics
(ii) keep metric fixed & consider different periodicities

Let's use (ii) & see when we can use diffeos & Weyl to match periodicities & metric.

Consider two tori $\mathbb{C}/\Lambda$ & $\mathbb{C}/\Lambda'$ w/

$$\Lambda \left\{ \begin{array}{l} w \sim w+1 \\ w \sim w+\tau \end{array} \right.$$

$$ds^2 = dw d\bar{w}$$

$$\Lambda' \left\{ \begin{array}{l} w' \sim w'+1 \\ w' \sim w'+\tau' \end{array} \right.$$

$$ds^2 = dw' d\bar{w}'$$

When are these related by diffeo + Weyl?

$$\Lambda^1 \begin{cases} w' \sim w' + 1 \\ w' \sim w' + \tau \end{cases} \quad ds^2 = dw' d\bar{w}' \quad \text{? "no Weyl!"}$$

Consider diffeo

$$w' = a z + b \bar{z}$$

$$\begin{cases} w' \sim w' + 1 \\ w' \sim w' + \tau \end{cases} \Rightarrow \begin{cases} z \rightarrow z + 1 \Rightarrow a + b = 1 \\ z \rightarrow z + \tau \Rightarrow a\tau + b\bar{\tau} = 1 \end{cases}$$

$$w' = \frac{(\tau' - \bar{\tau})z + (\tau - \tau')\bar{z}}{\tau - \bar{\tau}} \quad \tau' = \tau \Rightarrow w' = z$$

$$dw' = \frac{(\tau' - \bar{\tau})dz + (\tau - \tau')d\bar{z}}{\tau - \bar{\tau}}$$

$$ds^2 = \frac{(\tau' - \bar{\tau})(\bar{\tau} - \bar{\tau}')}{|\tau - \bar{\tau}|^2} dz^2 + \frac{(\bar{\tau}' - \tau)(\tau - \tau')}{|\tau - \bar{\tau}|^2} d\bar{z}^2 + () dz d\bar{z}$$

cannot be undone
by Weyl rescaling.

$$ds^2 \neq e^{2\Omega} dz d\bar{z}$$

So unless $\tau = \tau'$, the w' torus is not conf. equivalent to the w torus!

$\Rightarrow$ τ is the modulus $\rightarrow 1\tau = 2\mathbb{R}$ parameters

Actually, not all τ are inequivalent, e.g.

$$\tau' = \tau + n, \quad n \in \mathbb{Z} \text{ are equivalent}$$

$$\text{since } \begin{matrix} w' \sim w' + 1 \\ w' \sim w' + \tau + n \end{matrix} \Leftrightarrow \begin{matrix} w' \sim w' + 1 \\ w' \sim w' + \tau \end{matrix}$$

Also, $\tau' = -\frac{1}{\tau}$ corresponds to swapping sides of torus (& rescaling)

$$\tau = ia \Rightarrow \tau' = \frac{-i}{a}$$

$$\begin{matrix} a \uparrow \\ \rightarrow \end{matrix}$$

$$\begin{matrix} \frac{1}{a} \downarrow \\ \rightarrow \end{matrix}$$

$$\begin{matrix} 1 \uparrow \\ \rightarrow \end{matrix} \xrightarrow{\text{scale}} \begin{matrix} a \uparrow \\ \rightarrow \end{matrix}$$

Together, $T: \tau \rightarrow \tau + 1$

$$S: \tau \rightarrow -\frac{1}{\tau}$$

$$\text{generate } \text{PSL}(2, \mathbb{Z}) \simeq \frac{\text{SL}(2, \mathbb{Z})}{\mathbb{Z}_2}$$

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d} \quad w | \quad ad - bc = 1$$

$$\tau \rightarrow \frac{a\tau + b}{c\tau + d}$$

$$w/ \quad ad - bc = 1$$

$$\& \quad a, b, c, d \in \mathbb{Z}$$

$$\& \quad (a, b, c, d) = (-a, -b, -c, -d)$$

$\Rightarrow$ Moduli space is $\frac{\text{UHP}}{\text{SL}(2, \mathbb{Z})}$

NB: We can use T to shift τ to live in

$$\text{Re } \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$\& S$ to map $|\tau| < 1 \rightarrow |\tau| > 1$

So the moduli space is the Fundamental Domain of $\text{SL}(2, \mathbb{Z})$

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} \mid \text{Im } \tau > 0, |\tau| > 1, \text{Re } \tau \in \left[-\frac{1}{2}, \frac{1}{2}\right] \right\}$$

