Recap:

- Infinitesimal moduli of metric: $P^{+} t_{(2)} = 0$
  - Well-defined holomorphic vector fields → CKV
  - " " 2-tensors → moduli
- Computed moduli of S^2 & T^2
- Computed CKG of S^2 & T^2
  - S^2: 3 + 3 CKV's → $PSL(2,\mathbb{C}) \sim SL(2,\mathbb{C}) / \mathbb{Z}_2$
    0 moduli
  - T^2: $U(1)\times U(1)$ CKV's
    1 $\mathbb{C}$ modulus → $\tau \in \mathcal{F} = \{\tau \in \mathbb{C} \mid \mathrm{Im}\,\tau > 0, |\tau| \geq 1, \mathrm{Re}\,\tau \in [-\tfrac{1}{2}, \tfrac{1}{2}]\}$
    since $\tau \sim \dfrac{a\tau + b}{c\tau + d}$ $PSL(2,\mathbb{Z})$

This was for S^2 $(g=0)$ & T^2 $(g=1)$. General story:

(V) Riemann-Roch Theorem

The Riemann-Roch Theorem relates the # of CKV's, moduli & the Euler number.

Let $\mu = \dim(\mathrm{Ker}\, P^{+}) = $ # of moduli
$\kappa = \dim(\mathrm{Ker}\, P) = $ # of CKV's

Then
$$\boxed{\mu - \kappa = \underbrace{-3\chi}_{\text{Euler number}} = 6g + 3b - 6}$$

& if $\chi > 0$ $\kappa = 3\chi$ & $\mu = 0$ no moduli
$\chi < 0$ $\kappa = 0$ & $\mu = -3\chi$ no CKV's

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max: $\chi = 2,\ g=0 \rightarrow S^2$ — CKG $= PSL(2,\mathbb{C}) \Rightarrow K=6$; no moduli $\Rightarrow \mu = 0$

$\chi = 0,\ g=1 \rightarrow T^2$ — CKG $= U(1)^2 \Rightarrow K=2$; modulus $\tau \Rightarrow \mu = 2$

$\chi = -2g,\ g>1 \rightarrow$ Higher genus surface — no CKG; moduli are complicated!

(vi) Amplitude revisited

We argued that the n-point amplitude is given by

$$A^{(n)}(\Lambda_i, p_i) = \sum_{\text{topologies}} g_s^{-\chi} \int \mathcal{D}X\, \underline{\mathcal{D}b\mathcal{D}c}\ e^{-S_{\text{poly}} - \underline{S_{\text{ghost}}}} \prod_{i=1}^{n} V_{\Lambda_i}(p_i)$$

This cannot be right.

For each metric modulus & CKV we have zero modes of b or c:

$$P^\dagger b_0 = 0 \iff P^\dagger t = 0$$

$$P c_0 = 0 \iff P\epsilon = 0$$

$$S_{\text{ghost}} = (b, Pc) = (P^\dagger b, c) = 0 \text{ for zero modes}$$

The zero modes drop out of action but because b & c are Grassmann-valued, the integral over zero modes $\mathcal{D}b_0$ & $\mathcal{D}c_0$ will make the amplitude vanish!

$\Rightarrow$ We need to include the right # of b & c insertions in amplitude so as to saturate the zero modes.

$\Rightarrow$ # of metric moduli $= \mu =$ # of b insertions in amplitude

# of CKV's $= K =$ # of c insertions in amplitude

$\boxed{\#\text{ of CKV's} = \kappa = \#\text{ of } c \text{ insertions in amplitude}}$

The b & c insertions can be computed from Faddeev–Popov procedure but taking into account:

(i) moduli of the metric

coordinates on moduli space

$$\delta \hat{g}^{(\xi\omega)}_{\alpha\beta}(m) = (\hat{P}\cdot\epsilon)_{\alpha\beta} + 2\omega\,\hat{g}_{\alpha\beta} + m^I \partial_I \hat{g}_{\alpha\beta}$$

$I = 1, \ldots, \mu$

think of $g_{\alpha\beta}(m)$ as function of moduli

$\Rightarrow \quad ds^2 = |d\sigma^1 + \underline{\tau}\, d\sigma^2|^2 \quad$ for T^2

(ii) fixing the position of κ Vertex operators to gauge fix the remnant gauge symmetry of the CKG

Doing this carefully, c.f. Weigand Ch. 5.3, Polchinski Ch. 5.3, Exercise Sheet 11 gives

integral over moduli space

usual F-P det.

$$A^{(n)}(\Lambda_i; p_i) = \sum_{\text{topologies}} g_s^{-\chi} \int \prod_{I=1}^{\mu} dm^I \int \underline{DX\,Db\,Dc\; e^{-S_{Poly} - S_{gh}}}$$

b insertions for each modulus

$$\times \prod_{I=1}^{\mu} \frac{1}{4\pi}(b, \partial_I \hat{g}) \underbrace{\prod_{i=1}^{\kappa} c(z_i)\bar{c}(\bar{z}_i) V_{\Lambda_i}(p_i, z_i, \bar{z}_i)}_{\text{unintegrated vertex operator}}$$

$$\times \underbrace{\prod_{i=\kappa+1}^{n} \int d^2 z_i\, V_{\Lambda_i}(p_i, z_i, \bar{z}_i)}_{\text{integrated vertex operators}}$$

integrated vertex operators $\leftarrow$ BRST invariant

Can be argued for from BRST invariance!

IV – Tree level amplitudes

IV - Tree-level amplitudes

Consider $n \geq 3$ point scattering @ tree level

$\Rightarrow$ Need to compute correlation function on S^2

We have $\kappa = 3$ CKV $\to PSL(2,\mathbb{C})$

$\mu = 0$ moduli

$\Rightarrow$ Use 3 unintegrated vertex operators

$$A^{(n)}_{tree}(\Lambda_i, p_i) = g_s^{-2} \langle c(z_1)\bar{c}(\bar{z}_1) V_{\Lambda_1}(p_1, z_1, \bar{z}_1) c(z_2)\bar{c}(\bar{z}_2) V_{\Lambda_2}(p_2, z_2, \bar{z}_2)$$
$$\times\, c(z_3)\bar{c}(\bar{z}_3) V_{\Lambda_3}(p_3, z_3, \bar{z}_3) \prod_{i=4}^{n} \int d^2 z_i\, V_{\Lambda_i}(p_i, z_i, \bar{z}_i) \rangle$$

$$\sim \langle c(z_1) c(z_2) c(z_3) \bar{c}(\bar{z}_1) \bar{c}(\bar{z}_2) \bar{c}(\bar{z}_3) \rangle_{gh}$$
$$g_s^{-2} \langle V_{\Lambda_1}(p_1, z_1, \bar{z}_1) V_{\Lambda_2}(p_2, z_2, \bar{z}_2) V_{\Lambda_3}(p_3, z_3, \bar{z}_3) \prod_{i=4}^{n} \int d^2 z_i\, V_{\Lambda_i}(p_i, z_i, \bar{z}_i) \rangle_{Poly}$$

We can choose z_1, z_2, z_3 as any 3 points on S^2 in order to gauge fix $PSL(2,\mathbb{C})$ remnant gauge symmetry.

Typically, we choose

$$z_1 = 0, \quad z_2 = 1, \quad z_3 = \infty$$

c.f. Exercise Sheet 10, Qu. 3

(i) Ghost correlators

We need to compute

$$\langle c(z_1) c(z_2) c(z_3) \rangle_{gh} \quad \& \text{ its c.c.}$$

We can use standard results about 3-pt functions of CFT's, or use holomorphicity to compute this.

CFT's, or use holomorphicity to compute this.
let's do the latter:

- $c(z_1)c(z_2)c(z_3)$ has no singularities (singularities are in b-c)
- should vanish when $z_1 \to z_2$, $z_1 \to z_3$, $z_2 \to z_3$

$$\Rightarrow \langle c(z_1)c(z_2)c(z_3)\rangle_{gh} = z_{12}z_{13}z_{23} \times \underbrace{f(z_1,z_2,z_3)}_{\text{holomorphic}}$$

w\ $z_{ij} = z_i - z_j$

$c(z)$ should be well-defined on S^2 & has weight -1
$\Rightarrow$ it can grow at most like z^2 as $z \to \infty$
(c.f. discussion of CKV)

But $z_{12}z_{13}z_{23}$ is already quadratic in z_1, z_2 & z_3

$$\Rightarrow \langle c(z_1)c(z_2)c(z_3)\rangle_{gh} = z_{12}z_{13}z_{23} \times \underbrace{\text{const.}}_{\text{we don't care about this!}}$$

$$\langle c(z_1)c(z_2)c(z_3)\bar{c}(\bar{z}_1)\bar{c}(\bar{z}_2)\bar{c}(\bar{z}_3)\rangle_{gh}$$
$$= \text{const.}\ |z_{12}|^2|z_{13}|^2|z_{23}|^2$$

(ii) Now let's compute the matter part of amplitude.
For simplicity, we'll compute only tachyon scattering amplitude.

This avoids complications from polarisation tensor etc. but we still learn essential lessons. Extension to other vertex operators is not difficult, c.f. Polchinski Ch. 6.2.

Tachyon vertex operator:

$$\boxed{V_T(p,z,\bar{z}) = g_s :e^{ip\cdot X(z,\bar{z})}:}$$

can be fixed requiring unitarity

We want to compute:

$$\left\langle \prod_{i=1}^{n} g_s :e^{ip_i\cdot X(z_i,\bar{z}_i)}:\right\rangle_{Poly} = g_s^n \int \mathcal{D}X \prod_{i=1}^{n} :e^{ip_i\cdot X(z_i,\bar{z}_i)}: e^{\frac{1}{2\pi\alpha'}\int d^2z\, X\partial\bar{\partial}X} \qquad (*)$$

This is just a Gaussian integral because the theory is free!

We can write $(*)$ as

$$Z[J] = \int \mathcal{D}X \exp\left[\int d^2z \left(\frac{1}{2\pi\alpha'} X\cdot\partial\bar{\partial}X + iJ\cdot X\right)\right]$$

$$\text{w/}\quad J^\mu(z,\bar{z}) = \sum_i p_i^\mu\, \delta(z-z_i, \bar{z}-\bar{z}_i)$$

Complete the square, taking into account zero modes of $\partial\bar{\partial}$.

$\Rightarrow$ Expand $X^\mu(z,\bar{z})$ in a complete basis of eigenmodes of $\partial\bar{\partial}$:

$$X^\mu(z,\bar{z}) = \sum_I x^\mu_I\, X^I(z,\bar{z})$$

$$\text{w/}\quad \partial\bar{\partial} X^I(z,\bar{z}) = -\omega_I^2\, X_I(z,\bar{z}) \quad \text{- Eigenfunctions}$$

$$\int d^2z\, X^I(z,\bar{z}) X^J(z,\bar{z}) = \delta_{II'} \quad \text{- orthonormality}$$

$$\int d^2z \; X^{-}(z,\bar{z}) X^{\sigma}(z,\bar{z}) = \delta_{II'} \qquad \text{- orthonormality}$$

$$\sum_I X_I(z,\bar{z}) X_I(z',\bar{z}') = \delta(z-z', \bar{z}-\bar{z}') \qquad \text{- completeness}$$

The zero-modes only appear in the $J \cdot X$ term
$\Rightarrow$ these give a δ-function:

$$\delta(J_0)$$

w\ $J_0^{\mu} = \int d^2z \; J^{\mu}(z,\bar{z}) X_0(z,\bar{z})$

$\rightarrow$ Gives $\delta(\sum_i p_i) \Rightarrow$ Momentum conservation

For the non-zero modes, we complete the square:

$$Z[J] = \int DX \exp\left[\int d^2z \left(\frac{1}{2\pi\alpha'} X \cdot \partial\bar{\partial} X + iJ \cdot X\right)\right]$$

zero modes removed

$$\sim \int DX \exp\left[\frac{\pi\alpha'}{2} \int d^2z \, d^2z' \; J(z,\bar{z}) \cdot (\partial\bar{\partial})'^{-1} J(z',\bar{z}')\right]$$

$$\delta(J_0)$$

$$\sim \int DX \exp\left[\frac{\pi\alpha'}{2} \int d^2z \, d^2z' \; J(z,\bar{z}) \cdot J(z',\bar{z}') \underbrace{G'(z,\bar{z}; z',\bar{z}')}_{\text{reduced Green's function w\ out zero modes}}\right]$$

$$\partial\bar{\partial} G'(z,\bar{z}; z',\bar{z}') = \delta^2(z-z', \bar{z}-\bar{z}') - X_0^2$$

Recall: $G(z,\bar{z}; z',\bar{z}') = \frac{1}{2\pi} \ln|z-z'|^2$

Applying this to tachyon amplitude:

$$\left\langle \prod_{i=1}^{n} g_s :e^{ip_i \cdot X(z_i,\bar{z}_i)}: \right\rangle_{Poly} = g_s^n \, \delta\left(\sum_{i=1}^{n} p_i\right) \exp\left(\frac{\alpha'}{2} \sum_{j<l} p_j \cdot p_l \ln|z_{jl}|^2\right)$$

→ no sum over $j=\ell$ due to using G' rather than G ⇒ removes singularity of Green's fct as $z_i \to z_j$!

$$= g_s^n \,\delta\Big(\sum_{i=1}^n p_i\Big) \prod_{j<\ell} |z_j - z_\ell|^{\alpha' p_j \cdot p_\ell}$$

All together:

Tachyons

genus of Ω

momentum conservation

$$\boxed{A^{(n)}_{\text{tree}}(T, p_i) \sim g_s^{n-2} \underbrace{|z_{12}|^2 |z_{13}|^2 |z_{23}|^2}_{\text{ghost contribution}} \delta\Big(\sum_{i=1}^n p_i\Big) \times \Big(\prod_{i=4}^n \int d^2 z_i\Big) \prod_{j<\ell} |z_j - z_\ell|^{\alpha' p_j \cdot p_\ell}}$$

(iii) 4-point amplitude

Consider 4-point amplitude ⇒ one integrated vertex operator

$$A^4_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim g_s^2 \, |z_{12}|^{2+\alpha' p_1 \cdot p_2} |z_{13}|^{2+\alpha' p_1 \cdot p_3} |z_{23}|^{2+\alpha' p_2 \cdot p_3}$$

$$\times \delta(p_1 + p_2 + p_3 + p_4)$$

$$\times \int d^2 z \, |z_1 - z|^{\alpha' p_1 \cdot p_4} |z_2 - z|^{\alpha' p_2 \cdot p_4} |z_3 - z|^{\alpha' p_3 \cdot p_4}$$

We use $PSL(2,\mathbb{C})$ to fix $z_1 = 0$, $z_2 = 1$, $z_3 = \lambda \to \infty$

$$A^{(4)}_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim g_s^2 \, \delta\Big(\sum_i p_i\Big) \int d^2 z \, |z|^{\alpha' p_1 \cdot p_4} |1 - z|^{\alpha' p_2 \cdot p_4}$$