Lecture 16

Wednesday, 27 January 2021 09:04

Recap:

- Riemann-Roch Theorem: topology related to moduli - CKV

$$-3\chi = \mu - \kappa$$

μ: # of moduli; κ: # of CKV

- We computed scattering amplitudes
  - → required further gauge fixing (remnant conf. symm) & including moduli
  - → this gives extra b & c ghost insertions (μ insertions of b, κ insertions of c)

- Computed tree-level amplitudes (correlation fct's on S^2)
  - → fix $PSL(2,\mathbb{C})$ symmetry by choosing 3 vertex op. insertions e.g. $z = 0, 1, \infty$

- Used holomorphicity to compute $\langle c(z_1) c(z_2) c(z_3) \rangle = z_{12} z_{23} z_{13} \times \text{const.}$

- Tachyon n-point amplitude → specialised to 4

Last time, we saw 4-point tachyon scattering amplitude

$$A^{(4)}_{\text{tree}}(T, p_1, p_2, p_3, p_4) \sim g_s^2 \, \delta\left(\sum_i p_i\right) \int d^2 z \, |z|^{\alpha' p_1 \cdot p_3} |1-z|^{\alpha' p_2 \cdot p_4}$$

The integral gives a Γ-function:

$$\int d^2 z \, |z|^{2a-2} \, |1-z|^{2b-2} = \frac{2\pi \, \Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)} \quad \text{w/ } \underline{a+b+c=1}$$

Γ-fct generalises $n!$

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The 4-point amplitude takes a particular nice form in terms of Mandelstam variables:

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_3)^2, \quad u = -(p_1+p_4)^2$$

w/ $s+t+u = -\sum_i p_i^2 = \sum_i M_i^2 = -\frac{16}{\alpha'}$

$$\boxed{A^{(4)} \sim g_s^2\, \delta\Big(\sum_i p_i\Big) \frac{\Gamma\left(-1-\frac{\alpha' s}{4}\right)\Gamma\left(-1-\frac{\alpha' t}{4}\right)\Gamma\left(-1-\frac{\alpha' u}{4}\right)}{\Gamma\left(2+\frac{\alpha' s}{4}\right)\Gamma\left(2+\frac{\alpha' t}{4}\right)\Gamma\left(2+\frac{\alpha' u}{4}\right)}}$$

This is the Virasoro-Shapiro Amplitude

This formula predates string theory & was studied because it has remarkable properties:

(a) It has interesting singularities due to exchange of ∞-ly many particles

(b) It is symmetric in s, t, u "worldsheet duality"

(c) It has good UV behaviour.

Let's elaborate.

(a) $\Gamma(z)$ has first order poles @ $z = -n$, $n \in \mathbb{Z}^+$

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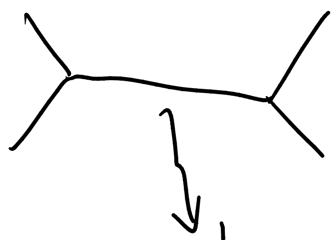
Consider fixed t & varying s. There are 1st order poles whenever

$$-1 - \frac{\alpha' s}{4} = -n, \quad n \in \mathbb{Z}^+$$

$$\Rightarrow s = \frac{4(n-1)}{\alpha'}, \quad n \in \mathbb{Z}^+$$

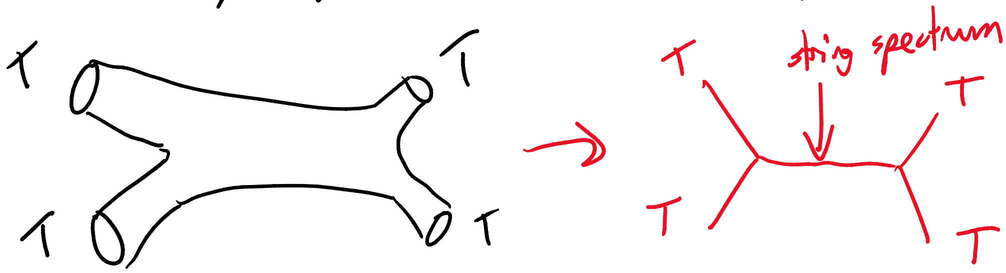


$\Rightarrow$ Exchange of particles of mass

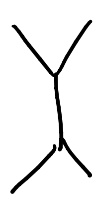
$$M^2 = \underbrace{\frac{4(n-1)}{\alpha'}}_{\text{string spectrum}}, \quad n \in \mathbb{Z}^+$$

$$M^2 = \frac{4}{\alpha'}(N-1)$$

This exactly reproduces the string spectrum!

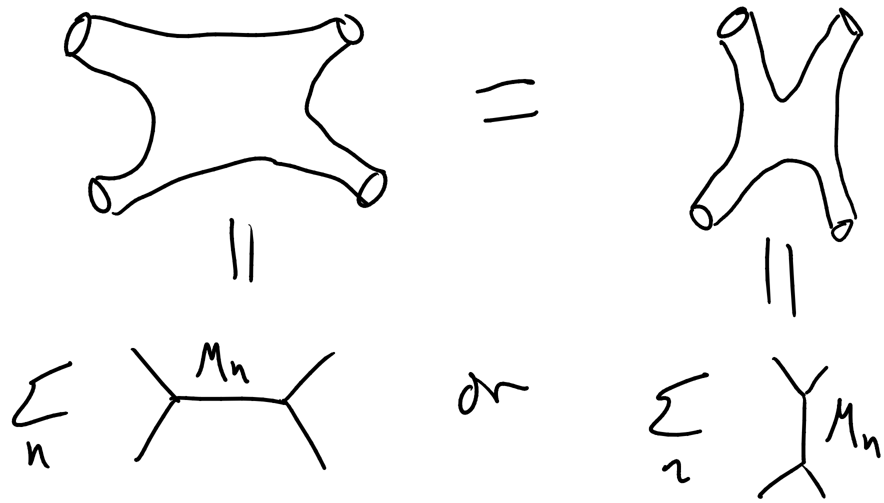


(b) We could have done the same thing w/ s fixed & varying $t \Rightarrow t$-channel exchange:

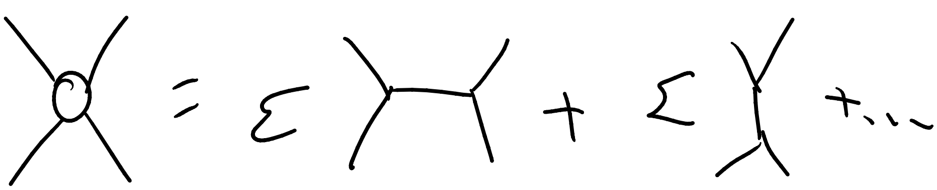
 in QFT

The Virasoro–Shapiro amplitude contains both

The Virasoro-Shapiro amplitude contains both s&t-channel exchanges, i.e. can be viewed as



While in QFT, we need to sum over s&t channels separately:



here we don't need to do s&t separately.

(c) For high energy scattering, consider $s \to \infty$, $t \to \infty$, $\frac{s}{t}$ fixed

hard-scattering limit

$\Rightarrow$ All exchanged momenta are at high energy, angle between incoming & outgoing is fixed:

$p_1 = \frac{\sqrt{s}}{2}(1, 1, 0, \ldots, 0)$ $\quad$ $p_3 = \frac{\sqrt{s}}{2}(1, \cos\theta, \sin\theta, 0, \ldots, 0)$

$p_2 = \frac{\sqrt{s}}{2}(1, -1, 0, \ldots, 0)$ $\quad$ $p_4 = \frac{\sqrt{s}}{2}(1, -\cos\theta, \sin\theta, 0, \ldots, 0)$

$p_2 = \frac{\sqrt{s}}{2}(1,-1,0,\dots,0)$ $\qquad$ $p_4 = \frac{\sqrt{s}}{2}(1,-\cos\theta,\sin\theta,0,\dots,0)$

Use $\Gamma(x) \sim \exp(x\ln x)$ as $x \to \infty$ (Stirling's Formula)

$$\Rightarrow \boxed{A^{(4)} \sim g_s^2\,\delta\Big(\sum_i p_i\Big)\exp\left[-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\right] \quad \text{as } s,t\to\infty,\ \frac{s}{t} \text{ fixed}}$$

Exponential fall-off of amplitude at high energies!

Scale of exponential fall-off is $\alpha' = \ell_s^2 \to$ length scale

i.e. exponential decay @ energies $E \gg \frac{1}{\ell_s} = \frac{1}{\sqrt{\alpha'}}$

This suggests the string is "fuzzy"/delocalised @ length scales $L \ll \sqrt{\alpha'}$

This is a <u>much</u> better behaviour than in QFT amplitudes: at best power-law decay, but often diverge, e.g. exchange particle of spin J

$$A \sim \frac{t^J}{s-M^2}$$

$\to \frac{1}{s}$ decay in s but divergent in t!

$\to$ Hard-scattering limit divergent!

We see that string is remarkably well-behaved in UV!

Even though string amplitudes contain an exchange of massive particles $M_n^2 = \frac{4}{\alpha'}(n-1)$, $n \in \mathbb{Z}^+$, the infinite sum massive particles $M_n^2 = \frac{4}{\alpha'}(n-1)$, $n \in \mathbb{Z}^+$, the infinite sum of these exchanges is much better behaved than any finite truncation!

Gravity & YM from string theory!

Knowing how the massless states scatter, we can derive an effective action at low energies $E\sqrt{\alpha'} \ll 1$, which reproduces the scattering amplitudes as $E\sqrt{\alpha'} \ll 1$.

Doing this for the gravitons show that at low energies, string theory produces GR:

$$S \sim \int d^{26}X \sqrt{G}\, R_{(26)}$$

+ terms involving Kalb-Ramond field (B), dilaton (ϕ).

We won't do this calculation, because it also requires expanding GR around Minkowski perturbatively.
Instead, we will later see a different way of recovering GR from string theory.

Similar calculations for open string show that at low energies, $E\sqrt{\alpha'} \ll 1$, we get amplitudes coincing w/ YM theory.

V - 1-loop amplitudes

Let's compute 1-loop amplitudes.
We will compute, for simplicity, only the vacuum amplitude. → Already contains interesting physics.

Use T^2: has $\mu = 2$ CKV's $(U(1) \times U(1))$
$\kappa = 2$ moduli $\to \tau \in \mathcal{F}$, fundamental domain

Need to compute T^2 correlators w\ 1 c & 1 $\bar{c}$ insertions & 1 b & 1 $\bar{b}$ insertions.

$$\Rightarrow A^{(0)}_{1\text{-loop}} \sim \int_{\mathcal{F}} d^2\tau \left\langle \frac{1}{4\pi}(b, \partial_\tau g) \frac{1}{4\pi}(b, \partial_{\bar\tau} g) \underbrace{\frac{c(z)\,\bar{c}(\bar{z})}{\text{vol}\, T^2}} \right\rangle_{T^2(\tau)}$$

required for BRST invariance: $\int d^2z \sim c\bar{c}$

BRST invar. $\int d^2z\, V^X$ & $c\bar{c}V^X$

From BRST perspective $\dfrac{c\bar{c}}{\underbrace{\int d^2z}_{\text{vol}\, T^2}} \to$ BRST invariant

$$(b, \partial_\tau g) = \int d^2z\, g^{\alpha\gamma} g^{\beta\delta}\, b_{\alpha\beta}\, \partial_\tau g_{\gamma\delta}$$

$$= \int d^2 z \, (b_{zz} \partial_\tau g_{\bar z \bar z} + b_{\bar z \bar z} \partial_{\bar\tau} g_{zz})$$

$$= \int d^2 z \, (b \, \partial_\tau g_{\bar z \bar z} + \bar b \, \partial_{\bar\tau} g_{zz})$$

Compute $\partial_\tau g_{\bar z \bar z}$ & $\partial_{\bar\tau} g_{zz}$.

Recall metric in real coordinates $g_{\alpha\beta}(\tau)$:

$$ds^2 = |d\sigma^1 + \tau \, d\sigma^2|^2$$

Defined $\mathbb{C}$ coordinates for fixed τ

$$\left.\begin{aligned} z &= \sigma^1 + \tau \sigma^2 \\ \bar z &= \sigma^1 + \bar\tau \sigma^2 \end{aligned}\right\} \Rightarrow ds^2 = dz \, d\bar z$$

Now consider $\tau \to \tau + \delta\tau$

$$ds^2(\tau + \delta\tau) = |d\sigma^1 + \tau \, d\sigma^2 + \delta\tau \, d\sigma^2|^2$$

$$= |dz + \delta\tau \, d\sigma^2|^2$$

But $\sigma^2 = \dfrac{z - \bar z}{\tau - \bar\tau} \Rightarrow d\sigma^2 = \dfrac{dz - d\bar z}{\tau - \bar\tau}$

$$\Rightarrow ds^2(\tau + \delta\tau) = \left| dz + \underline{\frac{\delta\tau}{\tau - \bar\tau}} (dz - d\bar z) \right|^2$$

$$\sim \underbrace{\left| 1 + \frac{\delta\tau}{\tau - \bar\tau} \right|^2}_{\text{infinitesimal}} \left| dz - \underbrace{\frac{\delta\tau}{\tau - \bar\tau}}_{\text{modulus}} d\bar z \right|^2$$

infinitesimal Weyl $\sim$ modulus dependence

$$\boxed{\begin{aligned} \Rightarrow \quad \partial_{\bar\tau} g_{zz} &= -\frac{1}{\tau-\bar\tau} = \frac{i}{2\,\mathrm{Im}\,\tau} \\ \& \quad \partial_\tau g_{\bar z\bar z} &= -\frac{i}{2\,\mathrm{Im}\,\tau} \end{aligned}}$$

$$\Rightarrow (b, \partial_\tau g) = -\frac{i}{2\,\mathrm{Im}\,\tau} \int d^2z\, \bar b(\bar z)$$

$$(b, \partial_{\bar\tau} g) = \frac{i}{2\,\mathrm{Im}\,\tau} \int d^2z\, b(z)$$

But b insertions picks up zero modes $\to$ constant on T^2

$\Rightarrow$ Can replace $\bar b(\bar z), b(z) \to \bar b(0), b(0)$ to get

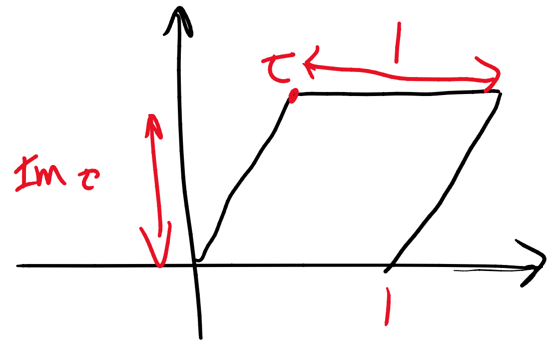
$$(b, \partial_\tau g) = \frac{-i}{2\,\mathrm{Im}\,\tau} \bar b(0) \int d^2z = \frac{-i\,\mathrm{vol}\,T^2(\tau)}{2\,\mathrm{Im}\,\tau}\, \bar b(0)$$

$$(b, \partial_{\bar\tau} g) = \frac{i\,\mathrm{vol}\,T^2(\tau)}{2\,\mathrm{Im}\,\tau}\, b(0)$$

All together:

$$A^{(0)}_{1\text{-loop}} \sim \int_F d^2\tau \left\langle b(0)\, \bar b(0)\, c(0)\, \bar c(0)\, \frac{\mathrm{vol}\,T^2}{(\mathrm{Im}\,\tau)^2} \right\rangle_{T^2(\tau)}$$

Compute $\mathrm{vol}\,T^2(\tau)$:



$$\mathrm{vol}\,T^2(\tau) = \mathrm{Im}\,\tau$$

$$\Rightarrow A^{(0)}_{1\text{-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \langle b(0)\,\bar{b}(0)\,c(0)\,\bar{c}(0)\rangle_{T^2(\tau)}$$

How do we compute this? Let's focus on the Polyakov part.

The Euclidean path integral w\ compactified time period ($\mathrm{Im}\,\tau$) gives partition function of theory @ temperature $T = \frac{1}{\mathrm{Im}\,\tau}$.

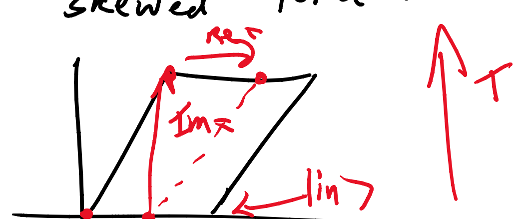
$$Z \sim \int dX\, e^{iS} \sim \sum_n \langle n| e^{-H\beta} |n\rangle$$

$$Z_X[\tau] = \underbrace{\mathrm{Tr}}_{\text{sum over all states}}\; e^{-2\pi(\mathrm{Im}\,\tau) H}$$

But we have a skewed torus!

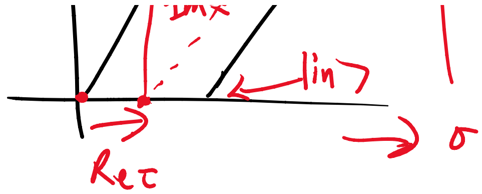
$$\langle \mathrm{out}| e^{iHT} e^{iPx} |\mathrm{in}\rangle$$

with $T \to i\,\mathrm{Im}\,\tau$, $x \to \mathrm{Re}\,\tau$



$\langle \text{out} | e^{\cdots} e^{\cdots} | \text{in} \rangle$

identify



$$Z_X[\tau] = \mathrm{Tr}\; e^{-2\pi(\mathrm{Im}\,\tau)H + 2\pi i(\mathrm{Re}\,\tau)P}$$

We have $P = L_0 - \bar{L}_0$, $H = L_0 + \bar{L}_0 - \frac{c+\tilde{c}}{24}$

Let's define $q = e^{2\pi i \tau}$, $\bar{q} = e^{-2\pi i \bar{\tau}}$

$$\boxed{Z_X[\tau] = \mathrm{Tr}\; q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \tilde{c}/24}}$$