Lecture 17

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<u>Recap:</u> • Calculate the Virasoro-Shapiro amplitude
(4-point tachyon tree level)

→ Nice properties:

(i) s,t,u symmetric

(ii) s-channel exchange & t-channel exchange

(iii) Good UV behaviour:

$$\exp\left(-\frac{\alpha'}{2}(s\ln s + t\ln t + u\ln u)\right)$$

hard scattering limit $s,t \to \infty$; $\frac{s}{t}$ fixed

Suggests string looks like delocalised objects @ $E \sim \frac{1}{\sqrt{\alpha'}}$ scales

(iv) Singularities in s-channel @ mass-shell

$$M^2_{(n)} = \frac{4}{\alpha'}(n-1)$$

• 1-loop vacuum amplitude ⇒ partition function

→ T^2 Riemann surface

→ 2 real moduli & 2 real conf. Kill vec's

⇒ $b, \bar{b}, c, \bar{c}$ insertions

→ Periodic Euclidean time ⇒ stat mech w\ temp. $T = \frac{1}{\text{Im}\,\tau}$

→ Compute this as a trace over states

$$A^{(0)}_{1\text{-loop}} = \int_{\mathcal{F}} \frac{d^2\tau}{\operatorname{Im}\tau} \underbrace{\langle b(0)\,\bar{b}(0)\,c(0)\,\bar{c}(0)\rangle_{T^2(\tau)}}_{Z_X[\tau]\;Z_{\text{ghost}}[\tau]}$$

$$Z_X[\tau] = \operatorname{Tr}\, q^{L_0 - c/24}\, \bar{q}^{\bar{L}_0 - \tilde{c}/24}$$

$$q = e^{2\pi i\tau}$$

For each state, we have

$$L_0 = \underbrace{\frac{\alpha' p^2}{4}}_{\text{momentum}} + \underbrace{N}_{\text{level}}, \qquad \bar{L}_0 = \frac{\alpha' p^2}{4} + \bar{N}$$

$$\operatorname{Tr}\, q^{L_0}\bar{q}^{\bar{L}_0} = \int \frac{d^{26}p}{(2\pi)^{26}} \operatorname{Tr}_{\text{osc.}} \langle p; N, \bar{N}|\, (q\bar{q})^{\frac{\alpha' p^2}{4}}\, q^{N}\bar{q}^{\bar{N}}\, |p; N, \bar{N}\rangle$$

$$= \int \frac{d^{26}p}{(2\pi)^{26}} \underbrace{\langle p|p\rangle}_{\text{Vol}_{\text{space-time}} \equiv V_{26}} e^{-\pi\alpha' p^2(\operatorname{Im}\tau)} \operatorname{Tr}_{\text{osc}} \langle N, \bar{N}|\, q^{N}\bar{q}^{\bar{N}}\, |N, \bar{N}\rangle$$

$$= V_{26}\int \frac{d^{26}p}{(2\pi)^{26}}\, e^{-\pi\alpha' p^2(\operatorname{Im}\tau)} \operatorname{Tr}_{\text{osc}} \langle N, \bar{N}|\, q^{N}\bar{q}^{\bar{N}}\, |N, \bar{N}\rangle$$

$$\sim V_{26}\,\frac{1}{\;\cdots\;}\, \operatorname{Tr}_{\text{osc}} \langle N, \bar{N}|\, q^{N}\bar{q}^{\bar{N}}\, |N, \bar{N}\rangle$$

$$\sim V_{26} \frac{1}{(\alpha'(\mathrm{Im}\,\tau))^{13}} \underbrace{\mathrm{Tr}_{osc} \langle N,\bar{N}| q^{N}\bar{q}^{\bar{N}} |N,\bar{N}\rangle}$$

consider $(\alpha_{-1})^i$ states: $1+q+q^2+\dots = \frac{1}{1-q}$

consider $(\alpha_{-2})^i$ states: $1+q^2+q^4+\dots = \frac{1}{1-q^2}$

$(\alpha_{-n})^i$ states: $1+q^n+q^{2n}+\dots = \frac{1}{1-q^n}$

$$\rightarrow \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^{26} \left(\prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n}\right)^{26} = \mathrm{Tr}_{osc} \langle N,\bar{N}| q^{N}\bar{q}^{\bar{N}} |N,\bar{N}\rangle$$

$$Z_X[\tau] = \frac{(q\bar{q})^{-\frac{26}{24}}}{(\alpha'\,\mathrm{Im}\,\tau)^{13}} \left(\prod_{n=1}^{\infty} \frac{1}{1-q^n}\right)^{26} \left(\prod_{n=1}^{\infty} \frac{1}{1-\bar{q}^n}\right)^{26} V_{26}$$

(i) Modular invariance

The above result can be rewritten in terms of the Dedekind η-function:

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$$

$$\Rightarrow Z_X[\tau] \sim \left(\sqrt{\mathrm{Im}\,\tau}\, |\eta(q)|^2\right)^{-26}$$ up to constants

The bc ghost contribution can be computed similarly (taking care of Grassmann-valuedness), see Polchinski Ch.7:

$$Z_{ghost}[\tau] \sim \left(|\eta(q)|^2\right)^2$$

$$\Rightarrow A^{(0)}_{1\text{-loop}} \sim \int_{\mathcal{F}} \frac{d^2\tau}{\mathrm{Im}\,\tau} \left(\sqrt{\mathrm{Im}\,\tau}\,|\eta(q)|^2\right)^{-26} \left(|\eta(q)|^2\right)^2$$

$$\boxed{\Rightarrow A^{(0)}_{1\text{-loop}} \sim \int_{\mathcal{F}} \overbrace{\frac{d^2\tau}{(\mathrm{Im}\,\tau)^2}}^{\text{measure}} \underbrace{\left(\sqrt{\mathrm{Im}\,\tau}\,|\eta(q)|^2\right)^{-24}}_{\text{integrand}}}$$

Recall that the torus modulus τ should be identified under $PSL(2,\mathbb{Z})$, generated by S & T transformations

$$S: \tau \to -\frac{1}{\tau} \quad \& \quad T: \tau \to \tau+1$$

$\Rightarrow$ The 1-loop partition function should be invariant as well.

This follows because η is a "modular form", having nice properties under $PSL(2,\mathbb{Z})$:

$$\eta(\tau+1) = e^{2\pi i/24}\,\eta(\tau)$$

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$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau)$$

$\Rightarrow$ $\mathrm{Im}\,\tau\,|\eta(\tau)|^4$ is invariant under S&T
$\Rightarrow$ invariant under $PSL(2,\mathbb{Z})$

Similarly, $\dfrac{d^2\tau}{(\mathrm{Im}\,\tau)^2}$ is invariant under $PSL(2,\mathbb{Z})$

c.f. Exercise Sheet 12.

$\Rightarrow$ Integrand & measure are both modular invariant.

The modular invariance of the 1-loop partition function is an important & useful consistency check for string theory. E.g. here it was crucial to have b, c ghosts & $D=26$.

(ii) UV-finiteness

To interpret the 1-loop partition function, compare to QFT.

Consider particle of mass m. The partition function will look like a sum over all particle paths will look like a sum over all 'particle' paths w/ topology of a circle (c.f. einbein action)

$$\Rightarrow Z_{S^1}(m^2) \sim V_{26}\int \frac{d^{26}p}{(2\pi)^{26}} \int_0^\infty \frac{d\ell}{2\ell}\, e^{-\ell\cdot \underbrace{\frac{1}{2}(p^2+m^2)}_{\text{Hamiltonian}}}$$

integral over moduli space, taking into account vol $S^1 \sim \ell$

$$\sim V_{26}\int_0^\infty \frac{d\ell}{\ell^{14}}\, e^{-m\ell^2}$$

diverges due to $\ell \to 0$ loops (corresponds to the usual $p\to\infty$ divergence if we had not integrated out p)

$\Rightarrow$ UV divergence.

By comparison, string theory 1-loop partition function is UV-finite. This is because the corresponding region

$\operatorname{Im}\tau \to 0$ is absent from the fundamental domain:

$$\mathcal{F} = \left\{ \tau \in \mathbb{C} \,\middle|\, \operatorname{Im}\tau > 0, |\tau| > 1, \operatorname{Re}\tau \in \left[-\tfrac{1}{2}, \tfrac{1}{2}\right] \right\}$$

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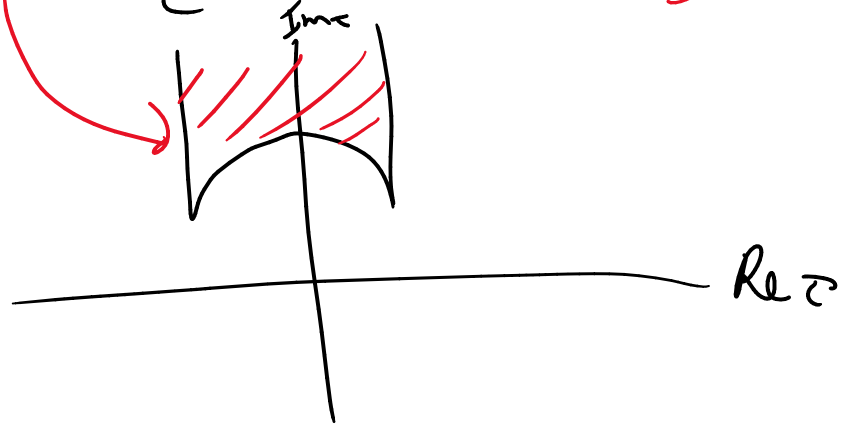

Therefore, modular invariance is the key reason for UV-finiteness of 1-loop partition function.

Again, we see that string theory has better UV-behaviour than QFT. Similar statements are believed to be true @ higher-loop order* but the moduli of (super-) Riemann surfaces make the calculations difficult!

* Proven @2-loop order.

<u>NB</u>: string theory is the only theory of quantum gravity & YM that is so well-behaved in UV!

(iii) IR divergence

The 1-loop partition function has an IR-divergence due to tachyons as $\mathrm{Im}\,\tau \to \infty$.

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$$Z \sim \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} |\eta(\tau)|^{-48} (\mathrm{Im}\,\tau)^{-12} \sim \int_{\mathcal{F}} \frac{d^2\tau}{(\mathrm{Im}\,\tau)^2} (\mathrm{Im}\,\tau)^{-12} |q\bar{q}|^{-1} \left| \prod_{n=1}^{\infty} (1-q^n) \right|^{-48}$$

$$\sim \int_{-\frac{1}{2}}^{\frac{1}{2}} d(\mathrm{Re}\,\tau) \int_A^\infty \frac{d\,\mathrm{Im}\,\tau}{(\mathrm{Im}\,\tau)^2} \frac{1}{(\mathrm{Im}\,\tau)^{12}} (q\bar{q})^{-1} (1+24q+\dots)(1+24\bar{q}+\dots)$$

terms like $q+\bar{q} \sim \sin 2\pi \mathrm{Re}\,\tau \times e^{-2\pi \mathrm{Im}\,\tau}$ integrate to 0

$$\sim \int_A^\infty \frac{d\,\mathrm{Im}\,\tau}{(\mathrm{Im}\,\tau)^2} \frac{1}{(\mathrm{Im}\,\tau)^{12}} \left((q\bar{q})^{-1} + (24)^2 + \dots \right)$$

$$\sim \int_A^\infty \frac{d\,\mathrm{Im}\,\tau}{(\mathrm{Im}\,\tau)^2} \frac{1}{(\mathrm{Im}\,\tau)^{12}} \Big(\underbrace{e^{4\pi \mathrm{Im}\,\tau}}_{\text{Tachyons}} + \underbrace{(24)^2}_{\text{Massless modes}} + \underbrace{\dots}_{\text{higher modes}} \Big)$$

Tachyons → Causes divergence!

Fortunately, this is an artifact of bosonic string theory ⇒ no need to worry!

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⑦ Low-Energy Effective Actions

I - Coupling to background fields

1 - Coupling to background fields

Let us now consider string theory moving in curved spacetimes.

Obvious generalisation of Polyakov action:

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{g}\, \underbrace{g^{\alpha\beta}}_{\text{worldsheet metric}} \partial_\alpha X^\mu \partial_\beta X^\nu \underbrace{G_{\mu\nu}(X)}_{\text{curved spacetime metric}}$$

(don't confuse w/ Einstein tensor!)

Actually, we can relate this back to what we've seen in scattering amplitudes. Consider $G_{\mu\nu}(X)$ as a perturbation of Minkowski spacetime:

$$G_{\mu\nu}(X) = \eta_{\mu\nu} + h_{\mu\nu}(X)$$

$$\Rightarrow Z_G = \int DXDg\, e^{-\frac{1}{4\pi\alpha'}\int d^2z\, \partial X \cdot \bar\partial X}$$

$$\times \underbrace{\exp\left(-\frac{1}{4\pi\alpha'}\int d^2z\, \partial X^\mu \bar\partial X^\nu h_{\mu\nu}\right)}_{\vee}$$

$$= \int DXDg\, e^{-\frac{1}{4\pi\alpha'}\int d^2z\, \partial X\cdot\bar\partial X}\left(1 - \frac{1}{\ldots}\int d^2z\, \partial X^\mu \bar\partial X^\nu h\ldots\right.$$

$$= \int \mathcal{D}X \mathcal{D}g \, e^{-\frac{1}{4\pi\alpha'}\int d^2z \, \partial X \cdot \bar{\partial} X} \left(1 - \underline{\frac{1}{4\pi\alpha'}\int d^2z \, \partial X^\mu \bar{\partial} X^\nu h_{\mu\nu}} + \ldots \right)$$

But now consider $h_{\mu\nu}$ corresponding to a plane wave w\ momentum p & polarisation $\zeta_{\mu\nu}$

$$\Rightarrow \quad V = \frac{1}{4\pi\alpha'} \int d^2z \, \zeta_{\mu\nu} \, \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X}$$

This is just the vertex operator of a massless graviton excitation!

Including e^V in the path integral corresponds to introducing a coherent state of gravitons

$\Rightarrow G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ consistent w\ being built from quantised string excitations (gravitons)!

We can reverse logic & couple the string to the other massless fields: $B_{\mu\nu}$ & ϕ

Kalb-Ramond form Dilaton

The B-field vertex operator is

$$V \propto \int d^2z \, \zeta \quad \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X}$$

$$V_B \sim \int d^2z \ \zeta_{[\mu\nu]} \partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X}$$

$$\Rightarrow S_{poly} \to S_{G,B,\phi} = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g} \Big(g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$

$$+ \underline{i} \, \underline{\epsilon^{\alpha\beta}} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X)$$

$$\epsilon^{\alpha\beta} = \pm \frac{1}{\sqrt{g}}$$

because of Euclidean signature

$$+ \underline{\alpha' \phi(X) R_{(2)}} \Big)$$

worldsheet Ricci Scalar

dilaton vertex operator w\ curved worldsheet metric see Polchinski