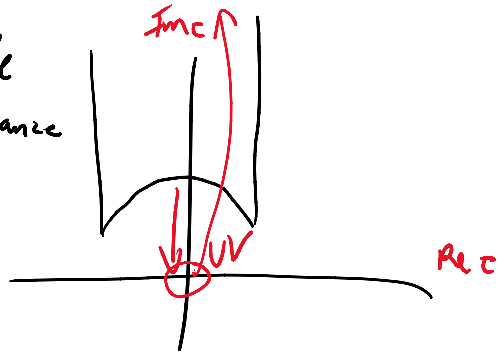
Lecture 18

Monday, 8 February 2021 09:06

Recap

- 1-loop string vacuum amplitude
  - → Modular invariance: $PSL(2,\mathbb{Z})$ invariance due to Dedekind η-function
  - → UV finite: don't integrate over $\operatorname{Im}\tau \to 0$
  - → IR divergence: due to tachyons as $\operatorname{Im}\tau \to \infty$



- Strings in background fields: curved spacetime, B-field, dilaton ϕ
  - → Minkowski metric → $G_{\mu\nu}$
  - → Include $\underbrace{\text{exponential of vertex operators}}_{\text{coherent state}}$

$$S_{poly} \to S_{G,B,\phi} = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\Big(g^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}(X)$$
$$+ i\epsilon^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu}(X) + \alpha' \phi(X) R_{(2)}\Big)$$

dimensional grounds: $[X] = -1$

Non-linear σ-model

Let's study the form of this action.

(i) Gauge symmetries

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This action is invariant under the gauge symmetries for G & B (recall BRST):

$$\delta G_{\mu\nu} = K^{\rho}\partial_{\rho} G_{\mu\nu} + G_{\mu\rho}\partial_{\nu} K^{\rho} + G_{\rho\nu}\partial_{\mu} K^{\rho}$$

$$\delta B_{\mu\nu} = \partial_{\mu} C_{\nu} - \partial_{\nu} C_{\mu}$$

c.f. Exercise Sheet 12, Qu 2

$B_{\mu\nu}$ forms a higher-dimensional a Maxwell theory:

$$\left.\begin{array}{l} A_{\mu} \to A_{\mu} + \partial_{\mu} \underbrace{\lambda}_{\text{scalar}} \\ \text{w/ invariant field strength} \\ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \end{array}\right\} \begin{array}{l} B_{\mu\nu} \to B_{\mu\nu} + \partial_{\mu} \underbrace{C_{\nu}}_{\text{1-form}} - \partial_{\nu} C_{\mu} \\ H_{\mu\nu\rho} = \partial_{\mu} B_{\nu\rho} + \partial_{\rho} B_{\mu\nu} + \partial_{\nu} B_{\rho\mu} \end{array}$$

c.f. Exercise Sheet 12, Qu 3

(ii) Dilaton & string coupling

Consider $\phi(X) = \lambda = \text{const.}$

Then the dilaton term in action becomes

$$\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, \alpha' \phi(X) R_{(2)}$$

$$= \frac{1}{4\pi} \lambda \int d^2\sigma \sqrt{g}\, R_{(2)}$$

$$= \lambda \underline{\chi} \xleftarrow{} \text{Euler characteristic}$$

$\Rightarrow$ Dilaton determines string coupling constant as

$$\underline{\underline{e^{-\lambda\chi}}} = (g_s)^{-\chi}$$

$$\langle \underline{\phi} \rangle = \ln g_s$$

constant mode

Usually, we choose asymptotic value as

$$\langle \phi \rangle = \phi_0 = \lim_{X \to \infty} \phi(X)$$

$$g_s = e^{\phi_0}$$

$\Rightarrow$ String coupling constant is not a free parameter. It's determined by background fields!

This is very different from QFT!

II - Weyl invariance

The string action must be Weyl invariant & we already discussed that this requires $D = 26$ spacetime dimensions when $B = \phi = 0$, $G = \eta$.

What happens in curved spacetime?

Recall that in flat spacetime, the worldsheet action was free after gauge-fixing the worldsheet metric, This allowed us to compute so much was free upto gauge-fixing the worldsheet metric, This allowed us to compute so much.

In presence of background fields, the string action is no longer free $\Rightarrow$ we will rely on perturbation theory on the worldsheet (not to be confused w/ spacetime perturbation theory controlled by g_s) to find requirement for Weyl invariance.

Consider $B = \phi = 0$ for simplicity.

$$\Rightarrow S = \frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu\, G_{\mu\nu}(X)$$

Expand around a simple classical solution

$$X^\mu(\sigma) = x_0^\mu + \underbrace{\sqrt{\alpha'}}_{}\, Y^\mu(\sigma)$$

dimensional reasons: $[X] = -1 = [\sqrt{\alpha'}]$
$\Rightarrow [Y] = 0$

Consider "small" fluctuations $Y^\mu \ll 1$.

Expand action around x_0:

$$G_{\mu\nu}(X)\, \partial X^\mu \cdot \partial X^\nu = \alpha'\Big[\underline{G_{\mu\nu}(x_0)} + \underline{\sqrt{\alpha'}\, G_{\mu\nu,\rho}(x_0)\, Y^\rho}$$
$$+ \underline{\frac{\alpha'}{2} G_{\mu\nu,\rho\sigma}(x_0)\, Y^\rho Y^\sigma} + \dots\Big]\, \underline{\partial Y^\mu \partial Y^\nu}$$

interaction terms w/ coupling

interaction terms w/ coupling constants $G_{\mu\nu,\rho}(x_0)$ for y^μ

$G_{\mu\nu,\rho\sigma}(x_0)$

$\vdots$

In generic curved spacetime, string theory has ∞-ly many interactions w/ coupling constants packaged into $G_{\mu\nu}(X)$.

We want to use perturbation theory

$\Rightarrow$ coupling constants must be small

let $\frac{\partial G}{\partial X} \sim \frac{1}{r_c}$ characteristic radius of curvature $[r_c] = -1$

$\Rightarrow$ Dimensionless effective coupling constant: $\frac{\sqrt{\alpha'}}{r_c}$

We can perform perturbation series on worldsheet for

$$\frac{\sqrt{\alpha'}}{r_c} \ll 1 \quad \Rightarrow \alpha'\text{-expansion}$$

NB! string theory has two perturbative expansions!

$\rightarrow$ α'-expansion on worldsheet

$\rightarrow$ g_s-expansion from space-time perspective

(i) β-function

Weyl invariance $\Rightarrow$ scale invariance

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In 2-d QFT, scale invariance $\Rightarrow$ Weyl invariance
(believed to be true in higher dimensions but no proof!)

$\Rightarrow$ We need to ensure that the theory is scale invariant

$\Rightarrow$ β-function must vanish!

$$\beta_{\mu\nu}(G) \sim \mu \frac{\partial G_{\mu\nu}}{\partial \mu} \overset{!}{=} 0$$

To compute the β-function, we need to regularise the theory (i.e. extract UV-divergence) & renormalise by adding counter terms.

$\Rightarrow$ Having vanishing β-function means that the counterterms must vanish.

Let's do this computation: pick Riemann normal coordinates around x_0 (i.e. $\Gamma(G) = 0$ but $\partial\Gamma(G) \neq 0$)

$$\Rightarrow G_{\mu\nu}(X) = \delta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\nu\kappa}(x_0) Y^\lambda Y^\kappa + O(Y^3)$$

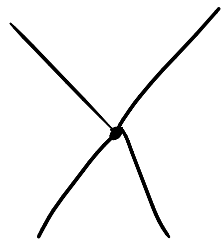
c.f. Exercise Sheet 13

$$\Rightarrow S = \frac{1}{4\pi} \int d^2\sigma \, (\partial_\alpha Y^\mu \partial^\alpha Y^\nu \delta_{\mu\nu} - \underbrace{\frac{\alpha'}{3} R_{\mu\lambda\nu\kappa} Y^\lambda Y^\kappa \partial_\alpha Y^\mu \partial^\alpha Y^\nu}_{\text{interacti...}}$$

interaction vertex

$+ \ldots)$

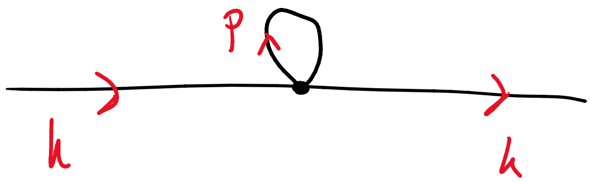
Quartic



$\sim R_{\mu\lambda\nu\kappa} \, k^{\mu}_{\alpha} k^{\nu\alpha}$

k^{μ}_{α} is 2-d worldsheet momenta

We want to compute β-function of $G_{\mu\nu}$ coupling, i.e. renormalised propagator. At 1-loop, we have UV-divergence from



gives contribution

$$\int \frac{d^2p}{(2\pi)^2} \underbrace{R_{\mu\lambda\nu\kappa} (k^{\mu} \cdot k^{\nu})}_{p\text{-independent}} \underline{\langle Y^{\lambda}(p) \, Y^{\kappa}(p) \rangle}$$

diverges as $p \to \infty$, c.f. position-space propagator

$$\langle Y^{\lambda}(\sigma) \, Y^{\kappa}(\sigma') \rangle \sim \delta^{\lambda\kappa} \ln |\sigma - \sigma'|^2$$

diverges as $\sigma \to \sigma'$

Use dimensional regularisation: $d = 2 + \epsilon$

$$\Rightarrow \int \frac{d^{2+\epsilon}p}{\ldots} \langle Y^{\lambda}(p) \, Y^{\kappa}(p) \rangle = \int \frac{d^{2+\epsilon}p}{(2\pi)^2} \frac{\delta^{\lambda\kappa}}{\ldots}$$

$$\Rightarrow \int \frac{d^{2+\epsilon}p}{(2\pi)^2} \langle y^\lambda(p)\, y^\kappa(p)\rangle = \int \frac{d^{2+\epsilon}p}{(2\pi)^2} \frac{\delta^{\lambda\kappa}}{p^2}$$

$$= \frac{\delta^{\lambda\kappa}}{2\pi} \int dp\, p^{\epsilon-1}$$

$$\sim \frac{1}{2\pi} \frac{\delta^{\lambda\kappa}}{\epsilon}$$

$\Rightarrow$ Loop-diagram contributes as

$$\int \frac{d^{2+\epsilon}p}{(2\pi)^2} R_{\mu\lambda\nu\kappa} (h^\mu \cdot h^\nu) \langle y^\lambda(p)\, y^\kappa(p)\rangle$$

$$\sim \frac{1}{2\pi} \frac{1}{\epsilon} R_{\mu\nu}\, h^\mu \cdot h^\nu$$

$\Rightarrow$ We need to add a counter-term

$$R_{\mu\lambda\nu\kappa}\, y^\lambda y^\kappa\, \partial y^\mu \cdot \partial y^\nu \rightarrow R_{\mu\lambda\nu\kappa}\, y^\lambda y^\kappa\, \partial y^\mu \cdot \partial y^\nu - \frac{1}{\epsilon} R_{\mu\nu}\, \partial y^\mu \cdot \partial y^\nu$$

This gives rise to the 1-loop β-function:

$$\boxed{\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} = 0}$$

String action in curved spacetime is only consistent (Weyl anomaly free) to 1st loop order on worldsheet

(Weyl anomaly free) to 1st loop order on worldsheet iff spacetime is Ricci flat

i.e. vaccuum Einstein equations are satisfied.

$$\boxed{\Rightarrow \text{Consistency of string action} \Longleftrightarrow \text{Einstein equations} \quad \left(@ \text{ lowest order in } \frac{\sqrt{\alpha'}}{r_c}\right)}$$

This is remarkable since we did not put in Einstein's equations by hand. They are required by consistency.

We can continue & compute the 2-loop β-function. This gives

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + \frac{(\alpha')^2}{2} R_{\mu\lambda\rho\sigma} R_\nu{}^{\lambda\rho\sigma} \overset{!}{=} 0$$

This way we can compute higher-derivative corrections (α'-corrections) to the Einstein equations in string theory.

(ii) Low-energy effective action

We can perform the β-function calculation including B & ϕ fields to get (at 1-loop):

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} = 0$$

$$\beta_{\mu\nu}(G) = \alpha' R_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu{}^{\lambda\kappa} = 0$$

$$\beta_{\mu\nu}(B) = -\frac{\alpha'}{2} \nabla^\lambda H_{\mu\nu\lambda} + \alpha' H_{\mu\nu\lambda} \nabla^\lambda \phi = 0$$

$$\beta_{\mu\nu}(\phi) = -\frac{\alpha'}{2} \nabla^2 \phi + \alpha' \nabla_\mu \phi \nabla^\mu \phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0$$

Since the β-functions control the Weyl anomaly, they appear in

$$\langle T^\alpha{}_\alpha \rangle = -\frac{1}{2\alpha'} \beta_{\mu\nu}(G) g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu$$

$$- \frac{i}{2\alpha'} \beta_{\mu\nu}(B) \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu$$

$$- \frac{1}{2} \beta(\phi) R_{(2)}$$

lowest order in $O(\sqrt{\alpha'}/r_c) \ll 1$

We can derive these β-functions from a low-energy effective action (spacetime action)

$$S_{LEEA} = \frac{1}{2\kappa_0^2} \int d^{26}X \sqrt{-G}\, e^{-2\phi} \Big(R - \frac{1}{12} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + 4 \partial_\mu \phi \partial^\mu \phi\Big)$$

$\kappa_0^2 \sim \ell_s^{24}$, $\ell_s = \sqrt{\alpha'}$

cf. Exercise Sheet 12, Qu. 4

This action agrees exactly w\ the action determined by scattering amplitudes to lowest order in g_s. This is a great consistency check for string theory!