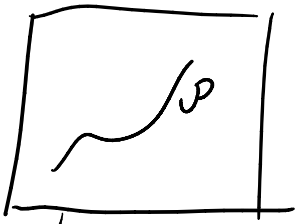
Lecture 2

Friday, 13 November 2020 08:52

Recap from lecture 1



Relativistic point particle

3 actions

(i) naive action w/ Poincaré symmetry

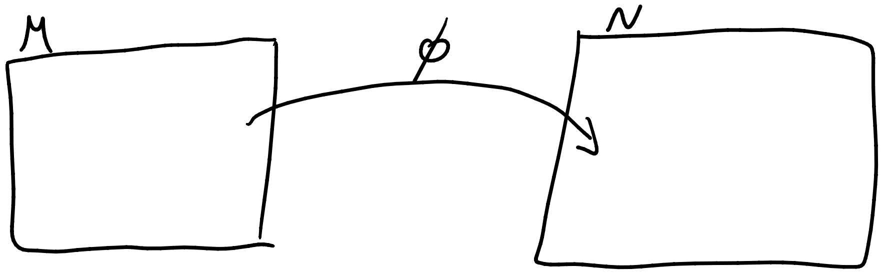
(ii) Nambu-Goto action $\quad S_{NG} = \int_{\mathcal{P}} d\tau \sqrt{-\underbrace{\dot{X}\cdot\dot{X}}_{\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}}$

- 1-dim diffeo $\tau \to \tilde{\tau}(\tau)$
- Poincaré symmetry
- Gauge fix $\to$ (i)

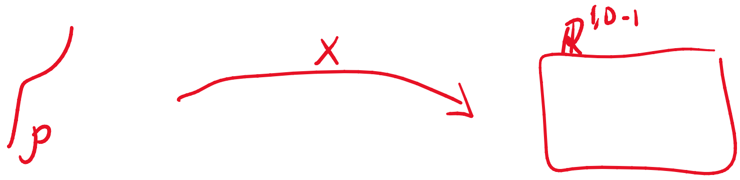
(iii) Polyakov action $\quad S_p = \frac{1}{2}\int_{\mathcal{P}} d\tau \left(e^{-1}\dot{X}^2 - e m^2\right)$

- Extra DoF $e(\tau) \to$ 1-dim GR
- Appears only algebraically
- Integrate e out $\to S_{NG}$
- Gauge fix $e \to$ better EoM
  $\to$ need to impose s.t.
  $$T \equiv \frac{\delta S_p}{\delta e} = 0$$

Pull-backs & Push-forwards



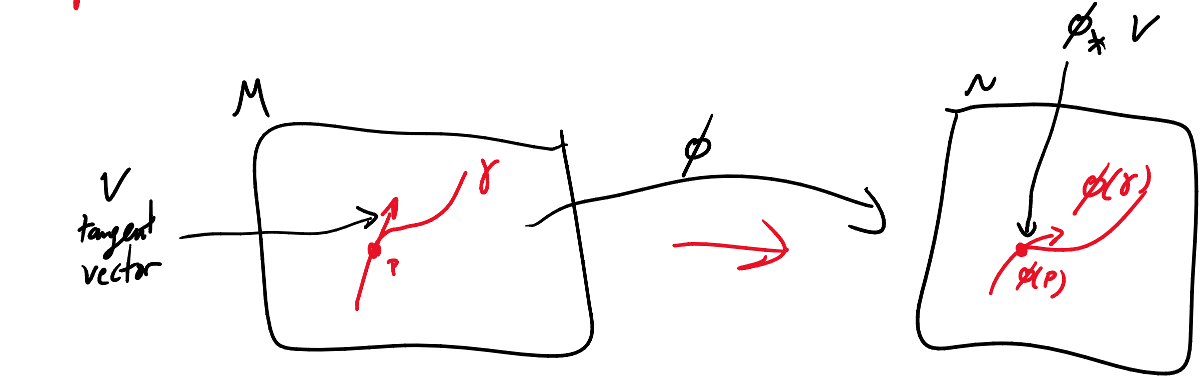
e.g.



"Surface elements" Covariant objects $\qquad$ co-vectors (1-form), metric

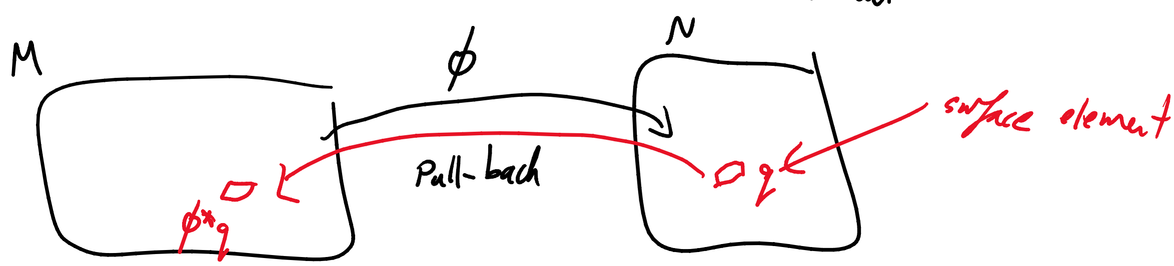
"Surface elements" Covariant objects — co-vectors (1-form), metric

"Slopes" Contravariant objects — vector (vector field)



Surface elements act on vectors

$\Rightarrow$ how $\perp$ is the vector to surface element

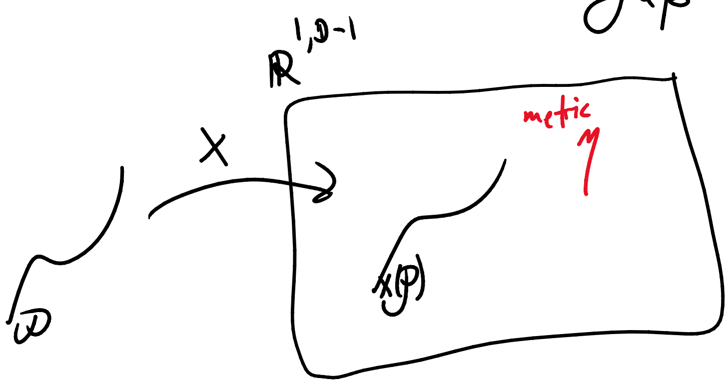


Look at $q(v)$: $\qquad \phi^* q(v) = q(\phi_* v)$

Metric: covariant $\quad g_{\mu\nu}$ on N

$\downarrow$ Pulled back

$$g_{\alpha\beta} = \underline{\partial_\alpha \phi^\mu}\, \underline{\partial_\beta \phi^\nu}\, g_{\mu\nu} \quad \text{on } M$$



What does η measure along $\mathcal{D}$?

$$\gamma = X^* \eta$$

$$\gamma_{\alpha\beta} = \underline{\partial_\alpha X^\mu}\, \underline{\partial_\beta X^\nu}\, \eta_{\mu\nu}$$

$$\underbrace{ds^2 = \eta_{\mu\nu}\, dy^\mu dy^\nu}_{\text{restrict to } \mathcal{D}}$$

$\to$ general in $\mathbb{R}^{1,D-1}$

$\longrightarrow$ general in $\mathbb{R}^{1,D-1}$

restrict to $\mathcal{P}$

$$ds_{\mathcal{P}}^2 = \boxed{\eta_{\mu\nu} \frac{dX^\mu}{d\tau} \frac{dX^\nu}{d\tau}} d\tau^2$$

Pull-back metric

II – Classical bosonic string action

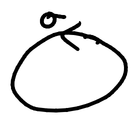
Particle $\rightarrow$ Worldline $\mathcal{P}$ in Minkowski space

String $\rightarrow$ Worldsheet Σ in Minkowski space

$\downarrow$ local coords

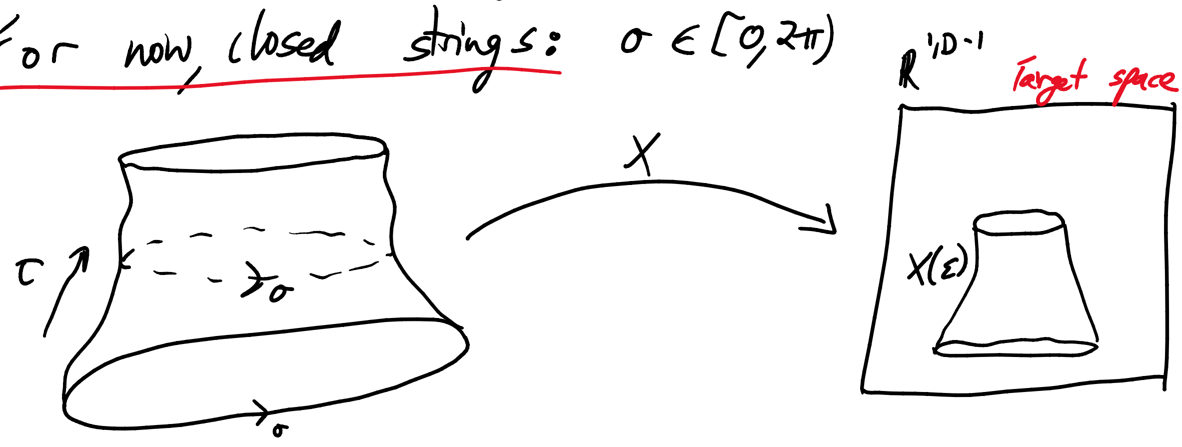
$\sigma^\alpha = (\tau, \sigma), \alpha = 0, 1$

Strings are 1-dim objects $\Rightarrow$ 2 kinds of strings possible (2 topologies of 1-d)

- Closed string  $\sigma \in [0, 2\pi)$

- Open string  $\sigma \in [0, \pi]$

<u>For now, closed strings:</u> $\sigma \in [0, 2\pi)$



$\Sigma \sim \mathbb{R} \times S^1$

Closed string B.C.'s: $X^\mu(\sigma, \tau) = X^\mu(\sigma + 2\pi, \tau)$

Nambu-Goto Action

Dynamical DoFs: $X^\mu(\sigma, \tau)$

Dynamical DoFs: $X^\mu(\sigma,\tau)$

Action: Recall point particle: $S_{NG} = -m\int_P ds = -m\int_P d\tau\sqrt{-\dot{x}\cdot\dot{x}}$

$$= -m\int_P d\tau\sqrt{X^*\eta}$$

minimises $\Delta s = \int_P ds$!

Analogous string action:

$$S_{NG} = -T\int_\Sigma dA = -T\int_\Sigma \underline{d^2\sigma}\sqrt{-\det\gamma}$$

$X^*\eta$

$\gamma = X^*\eta$ in local coords σ^α on Σ, & on $\mathbb{R}^{1,D-1}$

$$\boxed{\begin{aligned}\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu} &\equiv \partial_\alpha X^\mu\partial_\beta X^\nu\eta_{\mu\nu}\\ &\equiv \partial_\alpha X^\mu\cdot\partial_\beta X^\nu\end{aligned}}$$

is the pull-back metric on Σ

"induced metric"

Dimensions: $[X] = -1$, $[\sigma] = 0$ $(\sigma\sim\sigma+2\pi)$

$\Rightarrow [T] = 2 \Rightarrow$ Mass per unit length

$\Rightarrow T\sim$ tension

Often, $T = \dfrac{1}{2\pi\alpha'}$ $\qquad\alpha' =$ "alpha-prime"

$\Rightarrow [\alpha'] = -2$

Associate a length scale: "string length scale" or "string length"

$$\boxed{l_s = \sqrt{\alpha'}}$$

... string length

$$\boxed{l_s = \sqrt{\alpha'}}$$

Symmetries: (i) 2-d diffeos (local)

$$\sigma^\alpha \to \sigma'^\alpha(\sigma)$$

NB: shorthand $f(\sigma) = f(\tau, \sigma)$

w/ $X^\mu(\sigma)$ are scalar fields

(ii) Poincaré (Global)

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu$$

$$\Lambda_{(\mu\nu)} = 0$$

EOM: Use $\delta\sqrt{-\gamma} = \frac{1}{2}\sqrt{-\gamma}\,\gamma^{\alpha\beta}\delta\gamma_{\alpha\beta}$

Prove this assuming diagonalisability

$$\Rightarrow \quad \partial_\alpha\left(\sqrt{-\det\gamma}\;\gamma^{\alpha\beta}\partial_\beta X^\mu\right) = 0$$

Highly non-linear since $\gamma = X^*\eta$ depends on X!

"Wave equation" for X^μ

$$\nabla_\alpha \nabla^\alpha X^\mu = \Box_\gamma X^\mu = 0$$

but wrt metric γ that depends on X!

Gauge fixing: Can use worldsheet diffeos to fix "static gauge"

- $X^0 \equiv t = R\tau$

constant due to dimensions $[\sigma] = 0$, $[X] = -1$

Solve $\quad \frac{dX^0}{\dots} \to R: \tau'(\tau): \frac{dX^0}{\dots} = \frac{dX^0}{\dots}\frac{d\tau}{\dots} = R$

(Solve $\dfrac{dX^0}{d\tau} \to R$: $\tau'(\tau)$: $\dfrac{dX^0}{d\tau'} = \dfrac{dX^0}{d\tau}\dfrac{d\tau}{d\tau'} = R$)

dimensions $[\sigma] = 0$, $[X] = -1$

- NB: could also fix a Minkowski direction w\ σ, but not as useful

Let $X^\mu = (t, \underline{x})$

Consider a point in time when $\dfrac{d\underline{x}}{d\tau} = 0$

$\Rightarrow$ no kinetic energy at that time

Evaluate action for some small $d\tau$:

$$S = -T \int_\Sigma d\tau\, \underline{d\sigma}\; R \underline{\sqrt{\left(\frac{d\underline{x}}{d\sigma}\right)^2}}$$

$$= -T \int dt \times (\text{spatial length of string})$$

Kin. Energy $= 0 \Rightarrow S = -\int dt$ (Potential energy)

Potential energy $= \underset{\text{Energy/unit length = tension}}{\underline{T}} \times$ (spatial length of string)

Features:
- $\sqrt{}$ will cause problems w\ quantisation
- EoM complicated

Polyakov action

We have already seen a way to improve $\sqrt{}$

$\rightarrow$ couple 2-dim GR

Dynamical DoF's:

$$X^\mu(\sigma),\; g_{\alpha\beta}(\sigma)$$

signature $(-,+)$

only algebraically

Action:

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma\, \sqrt{-g}\; \underset{\text{inverse to}}{\underline{g^{\alpha\beta}}}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

$g = \det g_{\alpha\beta}$

$$S_p = \frac{\ }{2}\int_\Sigma d\sigma\, d\tau\, \sqrt{-g}\; g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \eta_{\mu\nu}$$

inverse to $g_{\alpha\beta}$

EoM's: $$\frac{\delta S_p}{\delta X^\mu} = 0 \Rightarrow \underline{\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\, \partial_\beta X^\mu\right) = 0}$$

$$\Rightarrow \Box_g X^\mu = 0$$

$$-\frac{\sqrt{-g}}{2} T_{\alpha\beta} \equiv \frac{\delta S_p}{\delta g^{\alpha\beta}}$$

$$= -\frac{T}{2}\sqrt{-g}\left(\underline{\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}} - \frac{1}{2}\underline{g_{\alpha\beta}}\, g^{\rho\sigma}\, \partial_\rho X^\mu \partial_\sigma X^\nu \eta_{\mu\nu}\right) = 0$$

Solution $$g_{\alpha\beta} = 2 f(\sigma)\, \underline{\partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}} \quad (+)$$

w/ $$f = \frac{1}{g^{\alpha\beta}\, \partial_\alpha X \cdot \partial_\beta X}$$

We see g EoM $\Rightarrow$ $g \propto X^* \eta$

But f drops out of EOM:

$$\partial_\alpha\left(\sqrt{-g}\, g^{\alpha\beta}\, \partial_\beta X^\mu\right) = \partial_\alpha\left(\sqrt{-\gamma}\, \gamma^{\alpha\beta}\, \partial_\beta X^\mu\right)$$

w/ $$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

Exercise: convince yourself of this! NB 2 dim very important!

$$g_{\alpha\beta} = 2 f \gamma_{\alpha\beta}$$

f also drops out of action!

$$g_{\alpha\beta} = 2 f \gamma_{\alpha\beta} \text{ in } S_p \Rightarrow S_{NG}\,!$$

Symmetries:

Symmetries:

(i) 2-d diffeos (local)

$$\sigma^\alpha \to \sigma'^\alpha(\sigma)$$

w\ X^μ scalar fields $\Rightarrow X^\mu(\sigma) \to X'^\mu(\sigma') = X^\mu(\sigma)$

$g_{\alpha\beta}$ worldsheet metric $\Rightarrow g_{\alpha\beta}(\sigma) \to g'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha}\frac{\partial\sigma^\delta}{\partial\sigma'^\beta} g_{\gamma\delta}(\sigma)$

Infinitesimally, $\sigma^\alpha \to \sigma^\alpha - \epsilon^\alpha \equiv \sigma'^\alpha$:

Exercise sheet 1
Problem 1

$$\delta X^\mu(\sigma) = \epsilon^\alpha \partial_\alpha X^\mu(\sigma)$$

$$\delta g_{\alpha\beta}(\sigma) = \nabla_\alpha \epsilon_\beta(\sigma) + \nabla_\beta \epsilon_\alpha(\sigma)$$

w\ $\nabla_\alpha \epsilon_\beta = \partial_\alpha \epsilon_\beta - \Gamma_{\alpha\beta}{}^\gamma \epsilon_\gamma$

Levi-Civita connection of worldsheet metric $g_{\alpha\beta}$

$$\Gamma_{\alpha\beta}{}^\gamma = \frac{1}{2} g^{\gamma\delta}\left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta}\right)$$

(ii) Poincaré symmetry (Global)

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + c^\mu, \quad \Lambda_{(\mu\nu)} = 0$$

(iii) Weyl (local)

$$g_{\alpha\beta}(\sigma) \to g'_{\alpha\beta}(\sigma) = \Omega^2(\sigma) g_{\alpha\beta}(\sigma)$$

Infinitesimally: $\Omega^2(\sigma) = e^{2\phi(\sigma)} = 1 + 2\phi(\sigma) + \ldots$

$$\delta g_{\alpha\beta}(\sigma) = 2\phi(\sigma) g_{\alpha\beta}(\sigma)$$

The Weyl symmetry is a new local symmetry & special to exactly 2 dimensions! Ω^2 drops out of action exactly like $f(\sigma)$ did when showing $S_P \to S_{NG}$.

when showing $S_P \to S_{NG}$.

Gauge fixing: Just like for point particle, we can use our local symmetries to fix $g_{\alpha\beta}$. We have:

- 2-dim diffeos $(\tau, \sigma) \to (\tau', \sigma')$
- Weyl symmetry $g \to \Omega^2 g$

$\Rightarrow$ 3 symmetries

But 2-dim metric $g_{\alpha\beta}$ has exactly 3 components!

$\Rightarrow$ Can use local symmetries to completely fix $g_{\alpha\beta}$

$\Rightarrow$ We can make $g_{\alpha\beta}$ (locally) flat!

Explicitly: (i) use worldsheet diffeos to set

$$g_{\alpha\beta} = e^{2\phi}\eta_{\alpha\beta} \qquad \text{conformal gauge}$$

(ii) use Weyl symmetry to set $\phi = 0$

$$g_{\alpha\beta} = \eta_{\alpha\beta} \qquad \text{flat gauge}$$

Caveat: locally!

Coord independent way of seeing this:

Can show, in 2 dim: $g_{\alpha\beta} \to e^{2\phi} g_{\alpha\beta} = g'_{\alpha\beta}$

$$\Rightarrow \sqrt{-g'}\,R' = \sqrt{-g}\,(R - 2\nabla^2\phi)$$

By solving $\nabla^2\phi = R$, can always perform Weyl rescaling s.t.

$$g' = e^{2\phi} g \Rightarrow R' = 0$$

But in 2 dim, Riemann tensor is completely specified by R:

by R:

$$R_{\alpha\beta\gamma\delta} = \frac{R}{2}\left(g_{\alpha\gamma}\, g_{\beta\delta} - g_{\alpha\delta}\, g_{\beta\gamma}\right)$$

$\Rightarrow$ $R'_{\alpha\beta\gamma\delta} = 0$ if $R' = 0$ $\Rightarrow$ metric g' is flat!

Can choose coordinates s.t. $g'_{\alpha\beta} = \eta_{\alpha\beta}$.

Gauge fixed EoM's & constraints:

Flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$

X EoM: $\Box_g X^\mu = \boxed{\partial_\alpha \partial^\alpha X^\mu = 0}$ Free scalar fields in 2 dim

g EoM: $T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\,\eta^{\delta\gamma}\,\partial_\delta X \cdot \partial_\gamma X = 0$

Explicitly:

$$\boxed{\begin{aligned} T_{01} &= \dot{X}\cdot X' = 0 \\ T_{00} = T_{11} &= \tfrac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0 \end{aligned}}$$

Constraints! $\dot{X} = \frac{\partial}{\partial \tau} X$, $X' = \frac{\partial}{\partial \sigma} X$

Features:

- Added 2-dim gravity $\Rightarrow$ got rid of $\sqrt{\ }$
- Extra DoF's $g_{\alpha\beta}$ appear algebraically
- Can integrate out or gauge fix

  $\swarrow$ recover S_{NG}

  $\Downarrow$ simple EoM's & constraints $T_{\alpha\beta} = 0$
- Get extra local Weyl symmetry

  $\Rightarrow$ Can make $g_{\alpha\beta} = \eta_{\alpha\beta}$ flat! (locally)