

Recap:

- BI action, effective one-loop open-string action
Expand in $\alpha' F \Rightarrow$ Maxwell
- DBI action including the scalar fields
Expand in $\alpha' F$ & $\alpha' \partial \phi \Rightarrow$ Maxwell + free massless scalar fields
- Tension of D-branes: $T \sim g_s^{-1} \Rightarrow$ non-perturbative
- DBI in closed string background fields:

$$S = -T \int d^{p+1} \xi \sqrt{-\det(\gamma_{ab} + B_{ab} + 2\pi\alpha' F_{ab})} e^{-\phi}$$

$$\gamma_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}$$

$$B_{ab} = \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}$$

- Non-abelian DBI unknown but leading order in $\alpha' F$ gives:
YM + scalar fields + potential

Multiple D-brane effective action:

$$S = - (2\pi\alpha')^2 T_p \int d^{p+1} \xi \text{Tr} \left(\frac{1}{4} F_{ab} F^{ab} + \frac{1}{2} D_a \phi^I D^a \phi^I - \frac{1}{4} \sum_{I \neq J} [\phi^I, \phi^J]^2 \right)$$

$$g_{\text{YM}}^2 \sim l_s^{p-3} g_s$$

The kinetic term for ϕ^I involves the gauge-covariant derivative:

$$D_\mu \phi^I = \partial_\mu \phi^I + [A_\mu, \phi^I]$$

derivative:

$$D_a \phi^I = \partial_a \phi^I + i [A_a, \phi^I]$$

Since ϕ^I is adjoint-valued.

The potential

$$V = -\frac{1}{4} \sum_{I \neq J} \text{Tr} [\phi^I, \phi^J]^2$$

is the most interesting part of the Lagrangian & can be found by direct computation or arguments based on T-duality / SUSY for superstring.

The interpretation of the scalars from the spacetime perspective as fluctuations becomes difficult since the ϕ^I 's don't commute.

$\Rightarrow$ D-branes see a non-commutative spacetime

The effective action reproduces the masses as D-branes are moved apart & associated Higgs mechanism that we saw from string spectrum in lecture 6.

In the superstring, the D_p -brane effective action includes fermionic excitations & gives rise to a super-Yang-Mills theory (SYM).

⑧ Circle compactification & T-duality

To make contact w/ real world, we need to compactify string theory to 4 dimensions. String theory

compactify string theory to 4 dimensions. String theory exhibits many interesting features when compactified, that differentiate it from point-particle theories.

One of these is T-duality when compactifying string theory on tori.

For simplicity, let us consider string theory on $\mathbb{R}^{1,24} \times S^1$

w/ S^1 coordinate $X^{25} \sim X^{25} + 2\pi R$

We will study string theory on this background from spacetime perspective & then from worldsheet perspective, where new features will become apparent.

I - spacetime perspective

(i) $L \gg R$

Let us consider the low-energy effective action for G, B, ϕ on $\mathbb{R}^{1,24} \times S^1$.

We can write the metric wlog as:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu + \underbrace{e^{2\sigma}}_{\downarrow} (dX^{25} + A_\mu dx^\mu)^2 \quad (*)$$

w/ $\mu, \nu = 0, \dots, 24$

$\downarrow$
VEV controls the radius of X^{25}

let us first only consider probing length scales

let us first only consider probing length scales
 $L \gg R$

$\Rightarrow$ All fields are independent of X^{25} .

Then $G_{\mu\nu}, A_\mu, \sigma$ really are 25-dimensional fields
 & inherit gauge transformations from 26-dim diffeos.

Consider 26-dim diffeo that only depends on 25-d
 coords:

$$(\delta X^M, \delta X^{25}) = (\Lambda^M, 1)$$

$$\text{w) } \frac{\partial \Lambda^M}{\partial X^{25}} = \frac{\partial 1}{\partial X^{25}} = 0$$

Then, c.f. Sheet 14, Qu. 2

$$\delta G_{\mu\nu} = \nabla_\mu \Lambda_\nu + \nabla_\nu \Lambda_\mu$$

$$\delta A_\mu = \partial_\mu \Lambda$$

$$\delta \sigma = 0$$

Levi-civita
connection
of $G_{\mu\nu}$

$G_{\mu\nu}$ is a 25-d
metric

A_μ is a 25-d
Maxwell field

σ is a 25-d
scalar field

We can plug (*) into the low-energy effective action
 to find dynamics of $G_{\mu\nu}, A_\mu, \sigma$.

26-dim Ricci scalar becomes

$$\mathcal{R}^{(26)} = \underset{\substack{\uparrow \\ D=25}}{\mathcal{R}} - 2e^{-\sigma} \nabla^2 e^\sigma - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu}$$

¹¹
 $D=25$
 Ricci scalar

$$\Rightarrow S = \frac{1}{2\kappa^2} \int d^{26}X \sqrt{-G^{(26)}} R^{(26)}$$

$$= \frac{2\pi R}{2\kappa^2} \int d^{25}X \sqrt{-G} e^{\sigma} \left(R - \frac{1}{4} e^{2\sigma} F_{\mu\nu} F^{\mu\nu} + \underline{\underline{\partial_{\mu}\sigma \partial^{\mu}\sigma}} \right)$$

strength of gravity
 weakened by vol of
 compactification

Reducing GR on S^1 gives lower-dim GR coupled to $U(1)$ gauge theory & 1 massless scalar field. This is known as Kaluza-Klein theory. But the massless scalar field does not fit observations. A realistic compactification needs a potential to "stabilise" (give mass) the scalar field. This is a typical problem of compactifications: there are massless scalars associated w/ overall volume & other deformations, known as moduli. These need to be given masses (moduli stabilisation).

Similarly, decomposing $B_{(2)}$, ϕ gives:

$$B_{(2)} = \frac{1}{2} B_{\mu\nu} dX^{\mu} dX^{\nu} + \tilde{A}_{\mu} dX^{\mu} dX^{25}$$

2-form in
 25-d

$$\phi = \phi$$

→ scalar in 25-d

→ vector in 25-d

25-d

$\phi = \varphi \rightarrow$ scalar in 25-d

$\sim$ vector in 25-d

The gauge transformations of $B_{(2)}$ reduce to gauge transformations of $B_{\mu\nu}$ & $U(1)$ gauge transformations of $\tilde{A}_\mu$.

$\Rightarrow$ At length scales $L \gg R$, we have:

25-dim metric: $G_{\mu\nu}$
25-dim 2-form: $B_{\mu\nu}$
2x 25-dim $U(1)$ gauge fields: A_μ
 $\tilde{A}_\mu$
2x 25-dim scalar fields: ϕ
 σ

(ii) Kaluza-Klein towers

let us now understand the X^{25} -dependence of fields.
for simplicity consider the dilaton ϕ c.f. sheet 14, Qn. 2

$$X^{25} \sim X^{25} + 2\pi R$$

$\Rightarrow$ Fourier Series:

$$\phi(X^\mu, X^{25}) = \sum_{n=-\infty}^{\infty} \phi_n(X^\mu) e^{inX^{25}/R}$$

W reality requiring $\phi_n^* = \phi_{-n}$

Plug this into the kinetic term

$$\int d^{24} X (\partial_\mu \phi \partial^\mu \phi + (\partial_{25} \phi)^2)$$

$$= 2\pi R \int d^{25} X \sum_{n=-\infty}^{\infty} (\partial_\mu \phi_n \partial^\mu \phi_{-n} - \frac{n^2}{R^2} |\phi_n|^2)$$

$\Rightarrow$ A single scalar field on $\mathbb{R}^{1,24} \times S^1$ gives rise to a Kaluza-Klein tower of infinitely many scalar fields on $\mathbb{R}^{1,24}$, labelled by $n \in \mathbb{Z}$. These have mass:

$$M_n^2 = \frac{n^2}{R^2}$$

As $R \rightarrow 0$, these fields become infinitely heavy & can all be integrated out except for $n=0$ mode, the X^{25} -independent mode. Thus for $L \gg R$, we recover the massless action seen before.

The Kaluza-Klein modes ϕ_n are charged under A_μ , the gauge field coming from the metric.

E.g. consider the diffeo $X^{25} \rightarrow X^{25} + \lambda(X^\mu)$,

$$\phi_n \rightarrow \phi_n e^{in\lambda/R}$$

ϕ_n has charge $\frac{n}{R}$.

All the other fields have similar KK tower of
 ... h.u. l.l. ... a bit more labels ...

All the other fields have similar KK tower of massive fields but require a bit more work due to gauge fixing, Higgs mechanism, etc!

The same KK tower exists for the tachyon:

$$M_n^2 = -\frac{4}{\alpha'} + \frac{n^2}{R^2}$$

$\Rightarrow$ For special values of the radius, $R = \frac{n}{2} \sqrt{\alpha'}$, the tachyon KK tower gives new massless scalar fields.

However, there is more going on in string theory, that we are missing so far.

II - Worldsheet perspective

Let us now study the worldsheet theory on $\mathbb{R}^{1,24} \times S^1$

There are two differences to $\mathbb{R}^{1,25}$:

(i) Momentum along X^{25} , p^{25} , is now quantised:

$$p^{25} = \frac{n}{R}, \quad n \in \mathbb{Z}$$

This ensures that the wavefunction is well-defined: $e^{ip \cdot X}$

(ii) String can wind the S^1 , so that the B.C. should be modified:

$$X^{25}(\sigma + 2\pi) = X^{25}(\sigma) + 2\pi R m$$

$$m \in \mathbb{Z}$$

m is the winding number.

We can now compute the string spectrum
c.f. Sheet 4, Qn 3(a)

$$X^{25}(\sigma, \tau) = x^{25} + \underbrace{\frac{\alpha' n}{R} \tau}_{\text{quantised } p^{25}} + \underbrace{mR\sigma}_{\text{accounts for winding}} + \underbrace{\text{oscillators}}_{\text{unchanged}}$$

let us now define

$$P_L = \frac{n}{R} + \frac{mR}{\alpha'} , \quad P_R = \frac{n}{R} - \frac{mR}{\alpha'}$$

s.t.

$$X^{25}(\sigma, \tau) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-)$$

W/

$$X_L^{25} = \frac{1}{2} (x^{25} + c^{25}) + \frac{1}{2} \alpha' P_L \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \hat{\alpha}_n^{25} e^{-in\sigma^+}$$

$$X_R^{25} = \frac{1}{2} (x^{25} - c^{25}) + \frac{1}{2} \alpha' P_R \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^{25} e^{-in\sigma^-}$$

The X^μ are unchanged, normal ordering

The X^μ are unchanged. normal ordering constant
 the $L_0 = \tilde{L}_0 = 1$ Virasoro constraint gives us
 the mass-shell condition & level-matching condition.
 From this we find the 25-dim. mass

$$M^2 = - \sum_{p=0}^{24} p_\mu p^\mu$$

becomes

$$M^2 = \underbrace{\frac{n^2}{R^2}}_{\substack{\downarrow \\ \text{KK tower} \\ \text{of excitations}}} + \underbrace{\frac{m^2 R^2}{\alpha'^2}}_{\substack{\downarrow \\ \text{energy due} \\ \text{to string wound} \\ \text{in times}}} + \frac{2}{\alpha'} (N + \tilde{N} - 2)$$

level from L.C. quantisation
normal ordering constant

& level-matching condition: $N - \tilde{N} = n \times m$

(i) Winding modes

$\Rightarrow$ on s' , we get

- KK tower of states w/ $\Delta M^2 = \frac{n^2}{R^2}$
- tower of winding states w/ $\Delta M^2 = \frac{m^2 R^2}{\alpha'^2}$

inherently "stringy"
 due to α' dependence

As $R \rightarrow \infty$, KK modes become light (continuous)
 Winding modes become heavy

Winding modes become heavy

← integrate
these out & obtain
25-dim. effective
action seen before.

As $R \rightarrow 0$, KK modes become heavy
winding modes become light (continuous)

↓
winding modes
cannot be ignored

When $R \ll \sqrt{\alpha'}$, the winding modes become light
& modify the effective action.