

Recap:

- Effective action for multiple D-branes:

$$V \sim \text{Tr} [\phi_x, \phi_y]^2$$

$\Rightarrow$ Non-commutative spacetime

- Compactification on S^1 from spacetime perspective:

$\rightarrow$ Two abelian gauge fields A_μ & $\tilde{A}_\mu$

$\rightarrow$ Two massless scalar fields

$\rightarrow$ Metric $G_{\mu\nu}$ & 2-form $B_{\mu\nu}$

- Considered string theory on $\mathbb{R}^{1,2,3} \times S^1$ & wrote down mode expansion on S^1

$\rightarrow$ KK tower $\Delta M^2 = \frac{n^2}{R^2}$ $\leftarrow$ can see from spacetime perspective

$\rightarrow$ winding tower $\Delta M^2 = \frac{m^2 R^2}{\alpha'^2}$ $\leftarrow$ cannot see from spacetime perspective

- As S^1 radius $R \rightarrow \infty$, KK tower becomes massless & continuous

winding tower becomes co-ly heavy

$R \rightarrow 0$, opposite

(ii) Massless sector (generic)

For generic values of R , the massless sector in $\mathbb{R}^{1,2,3}$ has (as usual)

$$m = n = 0$$

$$1 \leq \tilde{n} < \infty$$

$$m = n = 0$$

$$N = \tilde{N} = 1$$

& wrt $SO(1,24)$ have

$$\alpha_{-1}^M \tilde{\alpha}_{-1}^N |0;p\rangle \rightsquigarrow G_{\mu\nu}, B_{\mu\nu}, \phi \quad \begin{array}{l} \text{sym, traceless,} \\ \text{antisym \& trace} \\ \text{of } SO(1,24) \end{array}$$

$$(\alpha_{-1}^M \tilde{\alpha}_{-1}^{25} + \alpha_{-1}^{25} \tilde{\alpha}_{-1}^M) |0;p\rangle \rightsquigarrow A_\mu \text{ from metric}$$

$$(\alpha_{-1}^M \tilde{\alpha}_{-1}^{25} - \alpha_{-1}^{25} \tilde{\alpha}_{-1}^M) |0;p\rangle \rightsquigarrow \tilde{A}_\mu \text{ from 2-form}$$

$$\alpha_{-1}^{25} \tilde{\alpha}_{-1}^{25} |0;p\rangle \rightsquigarrow \sigma \leftrightarrow \text{radius of } S^1$$

These are the same massless fields that we identified from spacetime perspective.

(iii) Charges

We saw from spacetime perspective that KK modes are charged under A_μ . Let's see how to reproduce this from worldsheet & also investigate the winding modes.

We can compute the charge from the 3-point function:

$$\langle V_{\pm} V_{m,n} V_{-m,-n} \rangle$$

$\swarrow$ vertex operator of $A_\mu / \tilde{A}_\mu$ $\downarrow$ vertex operator w/ winding number m $\left. \vphantom{\begin{array}{c} \downarrow \\ \text{vertex operator} \\ \text{w/ winding number } m \end{array}} \right\} \text{tachyon}$

of $A_\mu / \tilde{A}_\mu$
 $\uparrow$
 V_+
 $\downarrow$
 V_-

w/ winding number m
 & KK number n } tachyon

$$V_{\pm}(p) \sim \int d^2 z \, \zeta_\mu (\partial X^\mu \bar{\partial} \bar{X}^{25} \pm \partial X^{25} \bar{\partial} \bar{X}^\mu) e^{ip \cdot X}$$

For $V_{m,n}$ we consider tachyon operator (level 0) for simplicity w/

$$P_L = \frac{n}{R} + \frac{mR}{\alpha'} \quad , \quad P_R = \frac{n}{R} - \frac{mR}{\alpha'}$$

$$\& V_{m,n}(p) \sim \int d^2 z \, e^{i p \cdot X} e^{i p_L X^{25} + i p_R \bar{X}^{25}}$$

25-d momentum

$$\Rightarrow \langle V_{\pm}(p) V_{m,n}(p_2) V_{-m,-n}(p_3) \rangle \sim \frac{\delta^{25}(p_1 + p_2 + p_3) \zeta_\mu (p_2^\mu - p_3^\mu)}{\times \frac{(p_L \pm p_R)}{\text{charge}}}$$

kinematical factor

$\Rightarrow$ KK modes are charged wrt A_μ w/ charge

$$q \sim P_L + P_R \sim \frac{n}{R}$$

winding modes are charged wrt $\tilde{A}_\mu$ w/ charge

$$\tilde{q} \sim P_L - P_R \sim \frac{mR}{\alpha'}$$

The winding mode charge is something new that we

1st L perspective

The winding mode change is something new that we are seeing from worldsheet perspective.

It arises because the string is charged under $B(2) \Rightarrow$ string winding S^1 is charged under the component

$$B_{\mu,25} \sim \tilde{A}_\mu$$

(iv) Extra massless fields & symmetry enhancement

For special values of R we get extra massless fields from KK modes of tachyon

$$R = \frac{n}{2} \sqrt{\alpha'} \Rightarrow M_m^2 = \frac{4}{\alpha'} \left(-1 + \frac{m^2}{n^2} \right)$$

n -th KK mode

$\Rightarrow n$ -th KK mode is massless!

This is also seen from spacetime perspective.

Additionally, when

$$R = \frac{2}{m} \sqrt{\alpha'} \Rightarrow M_m^2 = \frac{4}{\alpha'} \left(-1 + \frac{m^2 R^2}{\alpha'} \right)$$

$\Rightarrow m$ -th winding mode becomes massless!

This we missed when working directly in spacetime!

Something even more special happens when

$$R = \sqrt{\alpha'}$$

spectrum from: $R = \sqrt{\alpha'}$

spectrum from: $\frac{R = \sqrt{\alpha'}}$

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} (N + \tilde{N} - 2) = 0$$

w/ $N - \tilde{N} = m \times h$

Then we get massless fields:

- $N = \tilde{N} = 1$ & $m = h = 0$: usual massless sector for any $R \rightarrow G, \phi, B, \underbrace{A_\mu, \tilde{A}_\mu}_{U(1) \times U(1)}$ or

- Scalars {
- $N = \tilde{N} = 0$ & $n = \pm 2$: KK modes of tachyon
 $\sqrt{\alpha'} q = (\pm 2, 0)$ under $U(1) \times U(1)$
 - $N = \tilde{N} = 0$ & $m = \pm 2$: winding modes of tachyon
 $\sqrt{\alpha'} q = (0, \pm 2)$ under $U(1) \times U(1)$
 - $N = 1, \tilde{N} = 0$ & $n = m = \pm 1$: $\alpha_{-1}^\mu |0; p, n = m = \pm 1\rangle$ extra vectors
 $\alpha_{-1}^{25} |0; p, n = m = \pm 1\rangle$ extra scalar
w/ $\sqrt{\alpha'} q = (\pm 1, \pm 1)$ under $U(1) \times U(1)$
 - $N = 0, \tilde{N} = 1$ & $n = -m = \pm 1$: $\tilde{\alpha}_{-1}^\mu |0; p, n = -m = \pm 1\rangle$ extra vector
 $\tilde{\alpha}_{-1}^{25} |0; p, n = -m = \pm 1\rangle$ extra scalar
w/ $\sqrt{\alpha'} q = (\pm 1, \mp 1)$

We get 4 extra massless vectors which are charged under $U(1) \times U(1)$

$\Rightarrow$ The gauge group enhances to a non-abelian group:

$$U(1) \times U(1) \rightarrow SU(2) \times SU(2)$$

The 10 massless scalar fields form the

$$(3, 3) \oplus (1, 1)$$

of $SU(2) \times SU(2)$.

As σ gets a vev & we change value of R , the $SU(2) \times SU(2)$ breaks to $U(1) \times U(1)$ via the Higgs mechanism.

An action for these fields can be found by calculating amplitudes & reproduces the Higgs effect.

NB: This symmetry enhancement requires tachyon & so it does not occur in type II superstring. There is very close analogue in heterotic superstring theory.

III- T-duality

We have seen that KK modes & winding modes are very similar:

$$L = \frac{1}{R} \quad \dots \quad R = \frac{1}{L}$$

very similar:

- get light as $R \rightarrow \infty$
- get heavy as $R \rightarrow 0$

KK modes

- are charged under A_μ
- give extra massless scalars when $R = \frac{n}{2} \sqrt{\alpha'}$

- get heavy as $R \rightarrow \infty$

winding modes:

- get light as $R \rightarrow 0$
- are charged under $\tilde{A}_\mu$
- give extra massless scalars when $R = \frac{2}{n} \sqrt{\alpha'}$

looking at the spectrum

$$M^2 = \frac{n^2}{R^2} + \frac{m^2 R^2}{\alpha'^2} + \frac{2}{\alpha'} (N + \tilde{N} - 2)$$

We see spectrum is invariant under

$$m \leftrightarrow n \quad \& \quad R \leftrightarrow \frac{\alpha'}{R}$$

This is known as T-duality & extends to the full CFT $\Rightarrow$ all interactions are also invariant.

This means that because of string winding modes, strings don't see difference between circles of radius R & radius $\frac{\alpha'}{R}$ (w/ appropriate swap of KK vector $A \leftrightarrow \tilde{A}$ & dilaton $\star$).

R & radius $\bar{R}$ (with $\bar{R} = \alpha' / R$)
 $A_\mu \leftrightarrow \tilde{A}_\mu$ & dilaton ϕ).

Consider $R \rightarrow \infty$: winding modes become heavy & can ignore them @ low energies. The whole ∞ tower of KK modes becomes massless & continuous

$\Rightarrow$ signal in 25-d that a new direction in spacetime is opening up.

Now consider $R \rightarrow 0$: KK modes become heavy & can be integrated. The whole ∞ tower of winding modes becomes light & continuous & signals that a new 26th dimension is opening up.

The smallest radius that is seen by the string is the self-dual radius

$$R = \sqrt{\alpha'}$$

which under T-duality, $R \rightarrow \frac{\alpha'}{R}$, gets mapped to itself.

This also corresponds to radius of symm. enhancement.

(i) The new direction in spacetime

T-duality exchanges momentum & winding
 $n \leftrightarrow m$
 & $R \leftrightarrow \frac{\alpha'}{R}$

$$P_L = \frac{n}{R} + \frac{mR}{\alpha'} \Leftrightarrow P_L = \frac{n}{R} + \frac{mR}{\alpha'}$$

$\underbrace{\quad}_{n} \quad \underbrace{\quad}_{mR} \quad \underbrace{\quad}_{n} \quad \underbrace{\quad}_{mR}$

} a symmetric action on left/right-

$$P_R = \frac{n}{R} - \frac{mR}{\alpha'} \Leftrightarrow \textcircled{-} P_R = \frac{mR}{\alpha'} - \frac{n}{R} \quad \left. \vphantom{\frac{mR}{\alpha'}} \right\} \begin{array}{l} \text{action on} \\ \text{left/right-} \\ \text{moving} \\ \text{sectors} \end{array}$$

So define

$$Y^{25}(\tau, \sigma) = X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-)$$

Equivalently, $dX = \underbrace{\star}_{\text{worldsheet derivative}} dY$ $\swarrow$ worldsheet Hodge dual

(2-d version of E-M duality)

All OPE's of Y^{25} coincide w/ X^{25}

$\Rightarrow$ CFT is the same!

KK modes of X^{25} are winding modes of Y^{25}

Winding modes of X^{25} are KK modes of Y^{25}

(ii) Dilaton shift

We saw that under T-duality, $R \leftrightarrow \frac{\alpha'}{R}$, $A_\mu \mapsto \tilde{A}_\mu$.
The dilaton will also shift. This can also be seen by
e.g. looking @ LEEA.

Consider $R \gg \sqrt{\alpha'}$ sit. Winding modes can be ignored
for LEEA:

$$\Rightarrow S_{\text{LEE A}} = \frac{2\pi R}{2\ell_s^2 \alpha'^2} \int d^{25}X \sqrt{-G} e^{-2\phi} e^\sigma (R + \dots) \quad \begin{array}{l} \text{other massless} \\ \text{fields \&} \\ \text{KK modes} \end{array}$$

$$\Rightarrow S_{LEEA} = \frac{\alpha' R}{2\ell_s^2 g_s^2} \int d^{25} X \sqrt{-G} e^{-\alpha\phi} e^{\sigma} (R + \dots) \quad \text{KK modes}$$

Now consider $R \ll \sqrt{\alpha'}$, we should get same LEEA but w/ winding modes playing the role of KK modes in y^{25} direction:

$$S_{LEEA} = \frac{2\pi \alpha' / R}{2\ell_s^2 g_s'^2} \int d^{25} X \sqrt{-G} e^{\sigma} e^{-\alpha\phi} (R + \dots) \quad \text{winding modes as well}$$

These must agree:

$$g_s' = \frac{\sqrt{\alpha'}}{R} g_s$$

(iii) T-duality for open strings

We saw that T-duality exchanges

$$X^{25} \leftrightarrow Y^{25}$$

w/

$$dX^{25} = * dY^{25}$$

$$\Downarrow$$

$$\partial_\tau X^{25} = \partial_\sigma Y^{25}$$

$$\partial_\sigma X^{25} = \partial_\tau Y^{25}$$

Consider a D-brane $\perp$ to $X^{25} \Rightarrow$ Dirichlet BC's for $X^{25} \Rightarrow \delta X^{25} = 0$

$$\text{for } X^{25} \Rightarrow \delta X^{25} = 0$$

$$\Rightarrow \partial_\tau X^{25} = 0$$

T-dual direction has

$$\partial_\sigma X^{25} = 0$$

$\Rightarrow$ Neumann direction along y^{25}

Similarly Neumann along $X^{25} \Rightarrow$ Dirichlet along y^{25}

$\Rightarrow$ T-duality transverse to D_p -brane $\Rightarrow D(p+1)$ -brane

T-duality parallel to D_p -brane $\Rightarrow D(p-1)$ -brane

(iv) On T^d & superstring

T-duality of string theory on T^d forms the group $O(d, d; \mathbb{Z})$ which acts on

$$\begin{pmatrix} n^i \\ m_i \end{pmatrix} \begin{matrix} \leftarrow \text{KK modes} \\ \leftarrow \text{winding modes} \end{matrix}$$

in the fundamental:

$$\begin{pmatrix} n^i \\ m_i \end{pmatrix} \rightarrow \underset{\cap}{g} \begin{pmatrix} n^i \\ m_i \end{pmatrix}$$

$(d,d;\mathbb{Z})$

Superstring theory compactified on T^d also exhibits
T-duality which swaps IIA $\leftrightarrow$ IIB ($\& \text{SO}(32) \leftrightarrow E_8 \times E_8$
het.)

On more general compactifications, some similar phenomena
to T-duality occur. E.g. for Calabi-Yau compactifications
there is mirror symmetry which states that string
theory compactified on two topologically different
Calabi-Yaus is the same!

Further reading:

Superstring theory

BLT Ch. 7 - 9

Polchinski Ch. 10-12 (vol II)

GSW Ch. 4, 5, ... (mixed in
w/ other
topics)

Weigand Ch. 6

Uranga

<http://members.ift.uam-csic.es/auranga/firstpage.html>