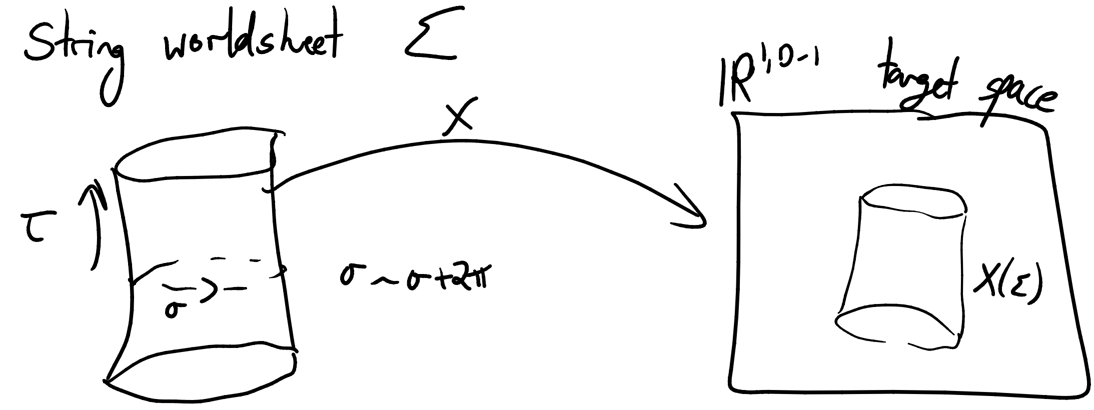
Lecture 3

Monday, 16 November 2020 08:58

Recap: String worldsheet Σ



$$S_{NG} = -T \int d^2\sigma \sqrt{-\det X^*\eta} \qquad (X^*\eta)_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} = \partial_\alpha X \cdot \partial_\beta X$$

$$S_P = -\frac{T}{2}\int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

tension $T = \frac{1}{2\pi\alpha'}$

Symmetries:
- Poincaré (G)
- Diffeomorphisms (L)
- Weyl (L)

$\Big\}$ 3 symmetries

(a) Integrate out $g \Rightarrow S_{NG}$

OR

(b) Gauge fix $g \Rightarrow g_{\alpha\beta} = \eta_{\alpha\beta}$ (locally) using Diffeos + Weyl

Constraints $\left.\frac{\delta S}{\delta g_{\alpha\beta}}\right|_{g=\eta} = 0 \Rightarrow T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X \overset{!}{=} 0$

EoM: $\partial_\alpha \partial^\alpha X^\mu = 0$

Today: (i) Interpret the constraints

(ii) Uniqueness of Polyakov action

III - Conformal symmetry

IV - Mode Expansion

III – Conformal Symmetry

IV – Mode Expansion

(i) Interpretation of constraints

Recall: $\partial_\alpha \partial^\alpha X^\mu = 0$ EoM

$$\left.\begin{aligned} \dot{X}\cdot X' &= 0 \\ \dot{X}^2 + X'^2 &= 0 \end{aligned}\right\} \text{Constraints} \qquad \dot{X} = \frac{\partial X}{\partial \tau}, \quad X' = \frac{\partial X}{\partial \sigma}$$

We still have residual gauge symmetry left:

Solve: $\dfrac{\partial X^0}{\partial \tau} = R, \quad \dfrac{\partial X^0}{\partial \sigma} = 0$

const. due to dimension

$$t \equiv X^0 = R\tau$$

Let $X^\mu = (t, \underline{x})$

EoM: $\ddot{\underline{x}} - \underline{x}'' = 0$ free wave equation

Constraints: $\dot{\underline{x}} \cdot \underline{x}' = 0 \rightarrow$ $\underbrace{\text{motion of string} \perp \text{string}}$ → transverse oscillations

& $\underbrace{\dot{\underline{x}}^2 + (\underline{x}')^2 = R^2}_{\text{interpretation for } R}$

Consider a point in time when $\dot{\underline{x}} = 0$

$$\Rightarrow \text{length of string} = \int_0^{2\pi} d\sigma \sqrt{\left(\frac{\partial \underline{x}}{\partial \sigma}\right)^2} = 2\pi R$$

R is just length of string when $\dot{\underline{x}} = 0$

(ii) Uniqueness of S_P

Could we have modified S_P?

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3 kinds of diffeo-invariant terms:

(a) $$S_1 = \frac{\lambda_1}{4\pi}\int_\Sigma d^2\sigma \sqrt{-\det g}\, R_{(g)}$$

Ricci scalar of g

cosmological constant

(b) $$S_2 = \frac{\lambda_2}{4\pi}\int_\Sigma d^2\sigma \sqrt{-\det g}$$

(c) $S_3 \to$ coupling to curved backgrounds

(a): $\int d^2\sigma \sqrt{-\det}\, R_{(g)}$ contains no dynamics: it is a topological invariant

$\Rightarrow$ it is invariant under continuous deformations of the metric, $g_{\alpha\beta}$!

Euler number $= \chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma d^2\sigma \sqrt{-\det g}\, R_{(g)}$

$\to$ we will see this later

(b): But EoM for $g_{\alpha\beta} \Rightarrow \lambda_2 = 0$ for consistency!

Exercise Sheet 2, Problem 5

(Also, spoil Weyl invariance)

(c): We'll see this at end of lectures

III - Conformal symmetry

Recall: we used Weyl symmetry & diffeos to fix $g_{\alpha\beta} = \eta_{\alpha\beta}$

BUT there is still some gauge symmetry left!

Consider diffeos which cause a scaling of metric:

Infinitesimally $\sigma^\alpha \to \sigma^\alpha - \epsilon^\alpha$

consider diffeos which

- Infinitesimally, $\sigma^\alpha \to \sigma^\alpha - \epsilon^\alpha$

w\ $$\delta g_{\alpha\beta} = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha \overset{!}{=} f(\epsilon)\, g_{\alpha\beta}$$

Consistency: $$\underbrace{g^{\alpha\beta}(\nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha)}_{2\nabla_\alpha \epsilon^\alpha} = f(\epsilon)\, \underbrace{g^{\alpha\beta} g_{\alpha\beta}}_{2}$$

$$\Rightarrow f(\epsilon) = \nabla_\alpha \epsilon^\alpha$$

Vector fields ϵ satisfying

$$\boxed{(P\cdot\epsilon)_{\alpha\beta} \equiv \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha - g_{\alpha\beta} \nabla_\gamma \epsilon^\gamma = 0}$$

$\Rightarrow$ ϵ causes a diffeo which scales metric

∂ ϵ is a Conformal Killing Vector Field

These induce conformal transformations

NB: these come from 2-dim diffeos i.e. change of coordinate

- We can undo the conformal transformations by an appropriate Weyl transformation: $\delta g_{\alpha\beta} = -(\nabla_\gamma \epsilon^\gamma)\, g_{\alpha\beta}$

$\ni$ Given any fixed $g_{\alpha\beta}$, combined action of conformal & Weyl transformations leaves $g_{\alpha\beta}$ invariant

$\ni$ We can still perform conformal transformations of σ^α w\ $g_{\alpha\beta}$ fixed e.g. $g_{\alpha\beta} = \eta_{\alpha\beta}$

- Take $g_{\alpha\beta}$ fixed ∂ the conformal symmetry becomes a global symmetry of the action generated by the conformal Killing vector fields:

$$\delta X^\mu = \epsilon^\alpha \partial_\alpha X^\mu$$

Conformal Killing vector

conformal Killing vector fields $(P\cdot\epsilon)_{\alpha\beta}=0$

e.g. ϵ^α s.t. $(P\cdot\epsilon)_{\alpha\beta}=0$

$$\epsilon^\alpha = \big(f(\sigma), g(\sigma)\big)\times\epsilon$$

$$\Rightarrow \delta X^\mu = \underbrace{\overset{\text{not local}}{\epsilon}}_{\text{global parameter}} \underbrace{\left(f(\sigma)\dot{X}^\mu + g(\sigma)X'^\mu\right)}_{\text{transf. law}}$$

Conserved currents of conformal transformations

For each conformal Killing vector field, we obtain a conserved current:

$$J_\epsilon^\alpha = T^{\alpha\beta}\epsilon_\beta \quad \forall\ P\cdot\epsilon=0$$

$$\nabla_\alpha J_\epsilon^\alpha = 0$$

Show: $\nabla_\alpha(T^{\alpha\beta}\epsilon_\beta)=0$ using $T^{\alpha\beta}g_{\alpha\beta}=0$

Exercise sheet 2, Problem

$$\nabla_\alpha(T^{\alpha\beta}\epsilon_\beta) = \underbrace{\nabla_\alpha T^{\alpha\beta}}_{=0 \text{ by cons of } T^{\alpha\beta}}\times\epsilon_\beta + \underbrace{T^{\alpha\beta}}_{\text{symm in } \alpha\leftrightarrow\beta}\underbrace{\nabla_\alpha\epsilon_\beta}_{\Rightarrow \text{sym in } \alpha\leftrightarrow\beta}$$

$$= \frac{1}{2}T^{\alpha\beta}(\nabla_\alpha\epsilon_\beta + \nabla_\beta\epsilon_\alpha)$$

using $P\cdot\epsilon=0$ $\Rightarrow \nabla_\alpha\epsilon_\beta+\nabla_\beta\epsilon_\alpha = g_{\alpha\beta}\nabla_\gamma\epsilon^\gamma$

$$= \frac{1}{2}T^{\alpha\beta}g_{\alpha\beta}\nabla_\gamma\epsilon^\gamma$$

$T^{\alpha\beta}g_{\alpha\beta}=0$ $\forall$ Weyl invariant

$$= 0$$

$= 0$

Weyl invariant theories

How many conformal Killing vectors are there?

In flat gauge, ∞-ly many : $\partial_\alpha \partial^\alpha \epsilon^\beta = 0$

$\Rightarrow$ Exercise Sheet 2 Problem 6

To see this let's consider

Light-cone coordinates : $\sigma^\pm = \tau \pm \sigma$

Flat 2-dim metric : $ds^2 = g_{\alpha\beta} d\sigma^\alpha d\sigma^\beta = -d\sigma^+ d\sigma^-$

i.e. $g_{++} = g_{--} = 0$

$g_{+-} = g_{-+} = -\frac{1}{2}$

NB: inverse metric $g^{++} = g^{--} = 0$

$g^{+-} = g^{-+} = -2$

$\Rightarrow V_+ = g_{+-} V^- = -\frac{1}{2} V^-$

Light-cone derivatives $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$

Measure: $d\tau d\sigma = \frac{1}{2} d\sigma^+ d\sigma^-$

Conformal Killing vectors: $(P \cdot \epsilon)_{\alpha\beta} = \nabla_\alpha \epsilon_\beta + \nabla_\beta \epsilon_\alpha - g_{\alpha\beta} \nabla_\gamma \epsilon^\gamma = 0$

What is the $(P \cdot \epsilon)_{++}$ equation?

$\partial_+ \epsilon_+ = 0 \Rightarrow \partial_+ \epsilon^- = 0$

$(P \cdot \epsilon)_{--} \Rightarrow \partial_- \epsilon_- = 0 \Rightarrow \partial_- \epsilon^+ = 0$

$$(P\cdot\epsilon)_{--} \Rightarrow \partial_- \epsilon_- = 0 \Rightarrow \partial_- \epsilon^+ = 0$$
$$(P\cdot\epsilon)_{+-} = 0$$

$\Rightarrow$ Conformal Killing vectors:

$$\boxed{\epsilon^+ = \underbrace{\epsilon^+(\sigma^+)}_{\text{Purely left-moving}} \quad \& \quad \epsilon^- = \underbrace{\epsilon^-(\sigma^-)}_{\text{Purely right-moving}}}$$

∞-ly many

Associated conf. coordinate transformations:

$$\boxed{\begin{aligned} \sigma^+ &\to \tilde\sigma^+(\sigma^+) \\ \sigma^- &\to \tilde\sigma^-(\sigma^-) \end{aligned}}$$

Conformal transformations

$$ds^2 = -d\sigma^+ d\sigma^- = -d\tilde\sigma^+ d\tilde\sigma^- \times \text{scaling}$$

Energy-Momentum Tensor:

$$T_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X - \tfrac{1}{2}\eta_{\alpha\beta}\eta^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X$$

$$T_{+-} = T_{-+} = 0 \iff T_{\alpha\beta}\eta^{\alpha\beta} = 0$$

True for any conf. theory

ES2, Problem 4

$$T_{++} = (\partial_+ X)\cdot(\partial_+ X)$$
$$T_{--} = (\partial_- X)\cdot(\partial_- X)$$

Conservation: $\nabla^\alpha T_{\alpha\beta} = 0 \Rightarrow \begin{aligned} \partial_- T_{++} &= 0 \\ \partial_+ T_{--} &= 0 \end{aligned}$

$$\Rightarrow \begin{aligned} T_{++} &= T_{++}(\sigma^+) \\ T_{--} &= T_{--}(\sigma^-) \end{aligned}$$

Conserved currents: $J_\epsilon^{\ \alpha} = T^{\alpha\beta}\epsilon_\beta$

Conserved currents: $J_\epsilon^{\alpha} = T^{\alpha\beta}\epsilon_\beta$

$$J_\epsilon^{+} = T^{++}\epsilon_+ = 4T_{--}\left(-\tfrac{1}{2}\right)\epsilon^{-} = -2T_{--}\epsilon^{-}$$

$$J_\epsilon^{-} = T^{--}\epsilon_- = -2T_{++}\epsilon^{+}$$

ϵ^+ & ϵ^- are independent:

$$J_{\epsilon^-}(\sigma^-) = -2T_{--}(\sigma^-)\epsilon^-(\sigma^-)$$

$$J_{\epsilon^+}(\sigma^+) = -2T_{++}(\sigma^+)\epsilon^+(\sigma^+)$$

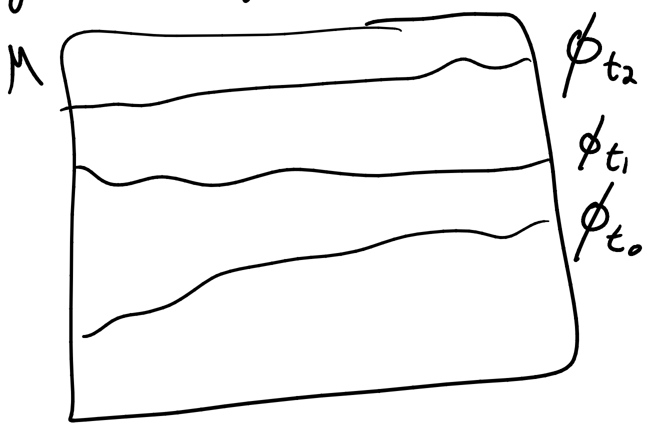
Conserved charges:

$$L_{\epsilon^+} = \frac{1}{2\pi\alpha'}\int d\sigma\ \epsilon^+ T_{++}$$

$$L_{\epsilon^-} = \frac{1}{2\pi\alpha'}\int d\sigma\ \epsilon^- T_{--}$$

generate conformal transformations via Poisson Bracket

Exercise Sheet 3



Conserved charge

$$Q_t = \int_{\phi_t} J^t$$

$$= \int_{\phi_t} d\underline{x}\ J^t$$

$$\frac{dQ_t}{dt} = 0$$

$$Q_{t_0} = Q_{t_1} = Q_{t_2}\ldots$$

Perform Fourier transform of T, gives Virasoro Generators

Virasoro Generators

$$\boxed{\begin{aligned} \tilde{L}_n &= \frac{1}{2\pi\alpha'}\int d\sigma\, T_{++}\, e^{in\sigma^+} \\ L_n &= \frac{1}{2\pi\alpha'}\int d\sigma\, T_{--}\, e^{-in\sigma^-} \end{aligned}}$$

equivalently: a basis of $\underbrace{\epsilon^+(\sigma^+)}_{\{e^{in\sigma^+}\}_{n\in\mathbb{Z}}}$, $\underbrace{\epsilon^-(\sigma^-)}_{\{e^{in\sigma^-}\}_{n\in\mathbb{Z}}}$

Exercise Sheet 3: these generate the Witt algebra

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}$$

$$\{\tilde{L}_m, \tilde{L}_n\}_{P.B.} = -i(m-n)\tilde{L}_{m+n}$$

IV – Mode Expansion

We've already seen $T_{\alpha\beta}$ in light-coordinates

$$\left.\begin{aligned} T_{++} &= (\partial_+ X)^2 = 0 \\ T_{--} &= (\partial_- X)^2 = 0 \end{aligned}\right\} \text{Constraints}$$

EoM: $\partial_\alpha \partial^\alpha X^\mu = 0 \Rightarrow \partial_+ \partial_- X^\mu = 0$

$$\Rightarrow X^\mu(\tau,\sigma) = \underbrace{X_L^\mu(\sigma^+)}_{\text{left-moving}} + \underbrace{X_R^\mu(\sigma^-)}_{\text{right-moving}}$$

Periodicity condition: $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$

Most general solution $\Rightarrow$ Fourier Expansion

Most general solution

$$(*)\begin{cases} X^\mu_L(\sigma^+) = \frac{1}{2}(x^\mu + c^\mu) + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-in\sigma^+} \\ X^\mu_R(\sigma^-) = \frac{1}{2}(x^\mu - c^\mu) + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-in\sigma^-} \end{cases}$$

NB: • normalisation chosen for convenience
→ x^μ, p^μ are c. of mass & mom. *Exercise Sheet 3*

• Periodicity : $X^\mu = X^\mu_L + X^\mu_R$

What condition is required on α^μ_n, $\tilde{\alpha}^\mu_n$ s.t. X^μ_L & X^μ_R are real?

• Reality of X^μ $\Rightarrow$ $\alpha^\mu_n = (\alpha^\mu_{-n})^*$

$$\tilde{\alpha}^\mu_n = (\tilde{\alpha}^\mu_{-n})^*$$

Virasoro Constraints:

We now need to impose $T_{\alpha\beta} = 0$ for $(*)$

But we saw that Virasoro generators are Fourier transform of $T_{\alpha\beta}$:

$$L_n = \frac{1}{2\pi\alpha'}\int d\sigma^- \, T_{--}\, e^{in\sigma^-} \overset{!}{=} 0$$

$$\tilde{L}_n = \frac{1}{2\pi\alpha'}\int d\sigma^+ \, T_{++}\, e^{in\sigma^+} \overset{!}{=} 0$$

$$\Updownarrow$$

$$T_{\alpha\beta} = 0$$

$T_{++} = (\partial_+ X)^2$ from $(*)$: $\partial_- X^\mu = \partial_- X^\mu_R = \sum_{n\in\mathbb{Z}} \alpha^\mu_n e^{-in\sigma^-}\sqrt{\frac{\alpha'}{2}}$

$T_{--} = (\partial_- X)^2$ where $\alpha^\mu_0 \equiv \sqrt{\frac{\alpha'}{2}}\, p^\mu$

$$T_{--} = (\partial_- X)^2$$

where $\alpha_0^\mu \equiv \sqrt{\frac{\alpha'}{2}}\, p^\mu$

$$\partial_+ X^\mu = \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in\sigma^+} \sqrt{\frac{\alpha'}{2}}$$

$$\tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$$

$$L_n = \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \; T_{--} e^{+in\sigma^-}$$

$$= \frac{1}{2\pi\alpha'} \int_0^{2\pi} d\sigma \, (\partial_- X)^2 e^{in\sigma^-}$$

$$= \frac{1}{2\pi\alpha'} \underline{\int_0^{2\pi} d\sigma} \sum_{m,k} \alpha_m \cdot \alpha_k \left(\frac{\alpha'}{2}\right) \underline{e^{i(n-m-k)\sigma^-}}$$

$$= \frac{1}{2\pi\alpha'} \frac{\alpha'}{2} \sum_{m,k} \alpha_m \cdot \alpha_k \, \delta_{n,m+k} \; 2\pi$$

$$T_{\alpha\beta} = 0 \left\{ \begin{array}{ll} \underline{L_n = \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{n-m} = 0} & \forall\, n \in \mathbb{Z} \\ \underline{\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m} = 0} & \forall\, n \in \mathbb{Z} \end{array} \right.$$

$\Rightarrow$ Give us constraints on string oscillator modes

$L_0 = \tilde{L}_0 = 0$ interesting:

$$L_0 = \frac{1}{2} \sum_m \alpha_m \cdot \alpha_{-m} = \frac{1}{2}(\alpha_0)^2 + \sum_{m>0} \alpha_m \cdot \alpha_{-m}$$

$$= \frac{1}{2} \frac{\alpha'}{2} p^2 + \sum_{m>0} \alpha_m \cdot \alpha_{-m}$$

But $p^2 = -M^2$ mass of particle in Minkowski space

$$M^2 = \frac{4}{\;} \sum \alpha_n \cdot \alpha_{-n} = \frac{4}{\;} \sum \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}$$

$$\boxed{M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}}$$

Mass-shell constraint & level-matching condition