Lecture 4

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Recap:
- Uniqueness of Polyakov Action
- Conformal symmetry by conf. Kill. vectors
- Virasoro generators as Fourier Transforms of $T_{\alpha\beta}$
- Mass-shell condition
- Mode expansion

generated Witt algebra

Conf. symmetry: In light-cone coords $\sigma^\pm = \tau \pm \sigma$

$$\epsilon^\alpha = \begin{pmatrix} \epsilon^+(\sigma^+) \\ \epsilon^-(\sigma^-) \end{pmatrix} \begin{matrix} \leftarrow \text{left-moving} \\ \leftarrow \text{right-moving} \end{matrix}$$

Virasoro generators: $J^\alpha = T^{\alpha\beta}\epsilon_\beta \rightarrow$ conserved for $\partial\cdot\epsilon = 0$

$$\tilde{L}_n = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, T_{++} e^{in\sigma^+}$$

$$L_n = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, T_{--} e^{in\sigma^-}$$

EoM: $\partial_+\partial_- X^\mu = 0$

$$X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

$$X^\mu_{L/R}(\sigma^\pm) = \underbrace{\frac{1}{2}(x^\mu \pm c^\mu)}_{\text{c.o.m. position}} + \frac{1}{2}\alpha' \underbrace{p^\mu}_{\text{c.o.m. momentum}} \sigma^\pm + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\underbrace{\tilde{\alpha}^\mu_n}_{\text{oscillators}} e^{-in\sigma^\pm}$$

Constraints:
$$\left.\begin{aligned} L_n &= \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m = 0 \\ \tilde{L}_n &= \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m = 0 \end{aligned}\right\} \alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\frac{\alpha'}{2}}\, p^\mu$$

$$L_0 = \tilde{L}_0 = 0 \Rightarrow M^2 = \frac{4}{\alpha'}\sum_{n>0}\alpha_n\cdot\alpha_{-n} = \frac{4}{\alpha'}\sum_{n>0}\tilde{\alpha}_n\cdot\tilde{\alpha}_{-n}$$

in quantum, operators $\Rightarrow$ ordering ambiguity!

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② Quantising the string

2) Quantising the string

We defined string theory as a 2-dim field theory on the worldsheet Σ, governed by Polyakov action:

$$S_P = -\frac{T}{2}\int_\Sigma d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\, \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

Quantise $\Rightarrow$ 2-dim QFT

Gauge symmetries: diffeos & Weyl

$\Rightarrow$ Constraints $T_{\alpha\beta} = 0 \iff L_n = \tilde{L}_n = 0$

Three ways in which we can & will proceed:

Canonical quantisation

OCQ • (Old) Covariant quantisation – quantise the system & impose constraints on Hilbert space after quantisation (c.f. Gupta-Bleuler quantisation for QED)

LCQ • Solve constraints classically first & quantise only physical states (c.f. Coulomb gauge in QED)

Path-integral quantisation

MCQ • Modern Covariant treatment of gauge theories using Faddeev-Popov ghosts

In string theory, each of these has problems but will accentuate different features:

<u>OCQ:</u> the QFT will be <u>non-unitary</u> due to negative-norm states <u>unless</u> we are in <u>$D=26$</u> spacetime dimensions.

<u>LCQ:</u> We will solve constraints in light-cone gauge. This breaks Lorentz symmetry of $\mathbb{R}^{1,D-1}$. The QFT will be manifestly unitary but will suffer

This breaks Lorentz symmetry of $\mathbb{R}^{1,D-1}$.
The QFT will be manifestly unitary but will suffer a Lorentz anomaly unless $D=26$.

MCQ: The QFT will develop a Weyl anomaly unless $D=26$.

We will be very brief w/ OCQ, push through to end w/ LCQ but not fully satisfactory.
We will then develop powerful CFT methods & use MCQ to the end.

I - Old Covariant Quantisation

Quantise D free scalars X^μ w/ $S_P = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X \cdot \partial^\alpha X$

& impose constraints $\dot{X}\cdot X' = \dot{X}^2 + X'^2 = 0 \Leftrightarrow L_n = \tilde{L}_n = 0$

(i) Quantise: $\Pi^\mu = -\frac{1}{2\pi\alpha'}\dot{X}^\mu$

& equal-time commutators:

$$[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = i\delta^\mu_\nu \delta(\sigma-\sigma')$$

$$[X^\mu(\sigma,\tau), X^\nu(\sigma',\tau)] = [\Pi_\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)] = 0$$

Using mode expansion:

$$[x^\mu, p_\nu] = i\delta^\mu_\nu, \quad [\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta^{\mu\nu}\delta_{m+n,0}$$

& all others vanishing

We see that:

- x & p get expected comm. relation for c.o.m. position/momentum
- α & $\tilde{\alpha}$ give ∞-ly many creation/annihilation operators of harmonic oscillator:

• α & $\tilde{\alpha}$ give ∞-ly many creation/annihilation operators of harmonic oscillator:

$$a^\mu_n = \frac{\alpha^\mu_n}{\sqrt{n}}, \quad a^{\dagger\mu}_n = \frac{\alpha^\mu_{-n}}{\sqrt{n}}, \quad n > 0 \text{ \& similarly for } \tilde{\alpha}^\mu_n$$

Ignore μ index for now: $[a_m, a^\dagger_n] = \delta_{m,n}$

$\Rightarrow$ SHO for each mode number!

Construct Fock space:

Vacuum $|0\rangle$ s.t. $\alpha^\mu_n |0\rangle = \tilde{\alpha}^\mu_n |0\rangle = 0$ for $n > 0$

This is not vacuum of spacetime but the vacuum of oscillations on a string!

The string vacuum also forms a representation of the x, p algebra:

$\Rightarrow$ Define $|0; p\rangle$ s.t.

$$\hat{P}^\mu |0; p\rangle = p^\mu |0; p\rangle$$

c.o.m. momentum operator

its eigenvalue

A generic state:

$$(\alpha^{\mu_1}_{-1})^{n_{\mu_1}} (\alpha^{\mu_2}_{-2})^{n_{\mu_2}} \cdots (\tilde{\alpha}^{\nu}_{-1})^{n_{\nu_1}} (\tilde{\alpha}^{\nu}_{-2})^{n_{\nu_2}} \cdots |0; p\rangle$$

Each state in Fock space $\rightarrow$ different excited state of string $\rightarrow$ different particles in space-time

Infinitely-many ways to excite $\Rightarrow$ ∞-ly many different particles in space-time

NB: At this stage there is no relation between p in $|0; p\rangle$

NB: At this stage, there is no relation between p in $|0;p\rangle$ & the excitations. The constraints will link these!

<u>Ghosts</u>: The Fock space norm is not positive-definite $\Rightarrow$ non-unitary QFT

$$[\alpha^\mu_n, \alpha^{\nu\dagger}_m] = n\,\eta^{\mu\nu}\,\delta_{n,m}$$

$\Rightarrow$ $|\psi\rangle = \alpha^0_{-k}|0;p\rangle$ has negative norm $(k>0)$

Show that

$$\langle\psi|\psi\rangle < 0$$

$$\begin{aligned}
\langle\psi|\psi\rangle &= \langle 0;p|\alpha^{0\dagger}_{-k}\,\alpha^0_{-k}|0;p\rangle \\
&= \langle 0;p|\alpha^0_k\,\alpha^0_{-k}|0;p\rangle \\
&= \langle 0;p|\cancel{\alpha^0_{-k}\alpha^0_k} + (-k)|0;p\rangle \\
&= -k\langle 0;p|0;p\rangle < 0 \leftarrow \text{ghosts!}
\end{aligned}$$

& similarly whenever we have odd # of timelike excitations.

$\rightarrow$ Like in QED in Lorentz

$\rightarrow$ Need to try & use gauge symmetry & constraints to save us.

<u>(ii) Constraints</u>:

Impose Virasoro constraints: $L_n = \tilde{L}_n = 0$

$$\left.\begin{aligned} \text{w/ } L_n &= \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m \\ \tilde{L}_n &= \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m \end{aligned}\right\} \text{classically}$$

c.f. Gupta-Bleuler in QED, we should require c.f. Gupta-Bleuler in QED, we should require

$$\langle phys' | L_n | phys \rangle = \langle phys' | \tilde{L}_n | phys \rangle = 0$$

Since $L_n^\dagger = L_{-n}$, we must impose:

$$L_n |phys\rangle = \tilde{L}_n |phys\rangle = 0 \quad \forall\ n > 0$$

But what are L_n in quantum theory?
Product of α's $\Rightarrow$ potential ordering ambiguity!

$$L_n = \frac{1}{2} \sum_m \alpha_{n-m} \cdot \alpha_m \quad \text{fine for } n \neq 0$$

But $L_0 = \frac{1}{2} \sum_m \alpha_{-m} \cdot \alpha_m \rightarrow \alpha_{-m}$ & α_m don't commute!

Let's define L_0 as normal ordered:

$$L_0 = \frac{1}{2} \alpha_0^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n$$

$$\tilde{L}_0 = \frac{1}{2} \tilde{\alpha}_0^2 + \sum_{n>0} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n$$

replace: $L_0 \rightarrow L_0 - a$

$\tilde{L} \rightarrow \tilde{L}_0 - a$ $\rightarrow$ constant!

Constraints: $(L_n - a\,\delta_{n,0}) |phys\rangle = 0$

$(\tilde{L}_n - a\,\delta_{n,0}) |phys\rangle = 0$ $\quad \forall\ n \geq 0$

NB: The quantum operators L_n generate the Virasoro algebra

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}$$

& similarly for $\tilde{L}_m$ c.f. Exercise Sheet 3, Problem 3

<u>Mass spectrum:</u> Recall L_0 involves $p^2 \Rightarrow$ related to mass
Constant a shifts the mass of physical states!

$$(L_0 - a)|phys\rangle = (\tilde{L}_0 - a)|phys\rangle$$

Use $L_0 = \sum_{n>0} \alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2$ w/ $\alpha_0^\mu = \sqrt{\frac{\alpha'}{2}}\, p^\mu$
& compute the mass

classical $\quad L_0^c = \frac{1}{2}\alpha_0^2 + \sum_n \alpha_{-n}\cdot\alpha_n = \frac{1}{2}\alpha_0^2 + \sum_n \alpha_n\cdot\alpha_{-n}$

quantum $\quad L_0^q = \frac{1}{2}\alpha_0^2 + \sum_n \alpha_{-n}\cdot\alpha_n$ or $\frac{1}{2}\alpha_0^2 + \sum_n \alpha_n\cdot\alpha_{-n}$
or anything inbetween

$$M^2 = \frac{4}{\alpha'}\left(-a + \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n\right)$$
$$= \frac{4}{\alpha'}\left(-a + \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\right)$$

Require ghosts to decouple $\Rightarrow a=1, D=26$
We won't do this here but see Green, Schwarz & Witten if interested

II - Light-cone quantisation

Now, let's first solve constraints classically & then quantise
We'll see that requiring Lorentz invariance in the quantum theory restricts:
$$a=1 \;\;\&\;\; D=26$$

We'll choose flat gauge $g_{\alpha\beta} = \eta_{\alpha\beta}$
Conformal symmetry: $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$

Conformal symmetry: $\sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma^{\pm})$
↳ use this to solve constraints

NB: EOM $\Rightarrow X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$ $2\times D$
Constraints $\Rightarrow (\partial_+ X)^2 = (\partial_- X)^2 = 0$ -2
Conf. symmetry $\Rightarrow \sigma^{\pm} \to \tilde{\sigma}^{\pm}(\sigma^{\pm})$ -2

Expect $\underbrace{2(D-2)}$ physical solutions
exactly # of transverse oscillations of string

Explicitly, use space-time light-cone coordinates

$$X^{\pm} = \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right)$$

s.t.

$$ds^2_{\text{spacetime}} = -2dX^+dX^- + \sum_{i=1}^{D-2} \underbrace{dX^i dX^i}_{\text{expect these to be the transverse oscillations}}$$

Let's gauge fix the conformal symmetry by taking

$$X^+ = x^+ + \alpha' p^+ \tau$$

Check that this is possible:
We want to do conf. transf.

$$\tau \to \tilde{\tau}(\tau, \sigma) \quad \& \quad \sigma \to \tilde{\sigma}(\tau, \sigma)$$

s.t.

$$\frac{\partial X^+}{\partial \tilde{\tau}} = \alpha' p^+ \quad \& \quad \frac{\partial X^+}{\partial \tilde{\sigma}} = 0$$

$$\frac{\partial X^+}{\partial \tilde\tau} = \alpha' p^+ \quad \& \quad \frac{\partial X^+}{\partial \tilde\sigma} = 0$$

$$\frac{\partial}{\partial \tilde\sigma_\pm} = \tilde\partial_\pm X^+ = \alpha' p^+$$

$$\tilde\partial_+ X^+ = \partial_+ X^+ \,\tilde\partial_+ \sigma^+ + \cancel{\partial_- X^+ \,\tilde\partial_+ \sigma^-}$$

0 since $\tilde\sigma^+ = \tilde\sigma^+(\sigma^+)$, $\tilde\sigma^- = \tilde\sigma^-(\sigma^-)$

$$\tilde\partial_- X^+ = \partial_- X^+ \,\tilde\partial_- \sigma^-$$

$\Rightarrow$ Need to solve:

$$\underbrace{\tilde\partial_+ \sigma^+}_{\text{fct of } \sigma^+} = \frac{\alpha' p^+}{\underbrace{\partial_+ X^+}_{\text{fct of } \sigma^+}} \quad \& \quad \underbrace{\tilde\partial_- \sigma^-}_{\text{fct of } \sigma^-} = \frac{\alpha' p^+}{\underbrace{\partial_- X^+}_{\text{fct of } \sigma^-}}$$

Works because:

$$\partial_+ X^+ = \partial_+ X_L^+(\sigma^+)$$
$$\partial_- X^+ = \partial X_R^+(\sigma^-) \qquad \checkmark$$

This is possible! By relabelling $(\tilde\tau, \tilde\sigma) \to (\tau, \sigma)$ we get:

$$\boxed{X^+ = x^+ + \alpha' p^+ \tau}$$

Light-cone gauge

The power of light-cone gauge comes when solving the constraints:

$$(\partial_+ X)^2 = -2(\partial_+ X^+)(\partial_+ X^-) + \sum_{i=1}^{D-2} (\partial_+ X^i)(\partial_+ X^i) = 0$$

$$(\partial_- X)^2 = -2\underbrace{(\partial_- X^+)}_{\text{know this}}\underbrace{(\partial_- X^-)}_{\text{solve for this}} + \sum_{i=1}^{D-2} (\partial_- X^i)(\partial_- X^i) = 0$$

know this

solve for this

physical DoF's

We now solve for X^-:

$$\partial_+ X_L^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i$$

$$\partial_- X_R^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i$$

In terms of mode expansion:

integration constants

$$X^-_{L/R}(\sigma^\pm) = \boxed{\frac{1}{2}(x^- \pm c^-)} + \frac{1}{2}\alpha' p^- \sigma^\pm + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^\pm_n e^{-in\sigma^\pm}$$

We find: $\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_m \sum_{i=1}^{D-2} \alpha^i_{n-m} \alpha^i_m \quad \forall n \in \mathbb{Z}$

& similarly for $\tilde{\alpha}_n^-$.

For α_0^- & $\tilde{\alpha}_0^-$:

$$(\dagger)\begin{cases} \dfrac{\alpha' p^-}{2} = \dfrac{1}{2p^+} \displaystyle\sum_{i=1}^{D-2} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n\neq 0} \alpha^i_n \alpha^i_{-n} \right) \\ \qquad = \dfrac{1}{2p^+} \displaystyle\sum_{i=1}^{D-2} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n\neq 0} \tilde{\alpha}^i_n \tilde{\alpha}^i_{-n} \right) \end{cases}$$

Mass-shell condition:

$$M^2 = 2p^+ p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i_{-n} \alpha^i_n$$

$$= \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}^i_{-n} \tilde{\alpha}^i_n$$

We have identified the physical excitations:

$x^i, p^i,\ \alpha_n^i,\ \tilde{\alpha}_n^i$ $\qquad x^+ \to$ can be absorbed into τ

$p^+, x^-,$ $\qquad p^- \to$ constrained by $(+)$

center of mass coord.

"transverse oscillations"