

Recap: How to quantise the string

- OCQ: quantise first $[X, \pi] = \delta$

- & impose the constraints à la Gupta-Bener $L_n |phys\rangle = 0 \quad \forall n \geq 0$
- we get negative norm states from the "timelike direction"
- normal ordering ambiguity in L_0 : $(L_0 - a) |phys\rangle = 0$
- $D=26$ & $a=1 \Rightarrow$ ghosts decouple

- LCQ: constraints first, then quantise

- breaks Lorentz symmetry (not manifest)
- No Lorentz anomaly $\Leftrightarrow D=26$ & $a=1$

- MCQ . Path-integral $\Rightarrow$ will be done later

In light-cone gauge: $X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^{D-1})$

& we can fix $X^+ = x^+ + \alpha' p^+ \tau$

$\Rightarrow$ Solve constraints: $\Rightarrow X^-$

$$(+) \left\{ \begin{aligned} \alpha_n^- &= \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_m \alpha_{n-m}^i \alpha_m^i \Rightarrow \frac{\alpha' p^-}{2} = \frac{1}{2p^+} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n \neq 0} \alpha_n^i \alpha_{-n}^i \right) \\ \tilde{\alpha}_n^- &= \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_m \tilde{\alpha}_{n-m}^i \tilde{\alpha}_m^i \Rightarrow \frac{\alpha' p^-}{2} = \frac{1}{2p^+} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n \neq 0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \right) \end{aligned} \right.$$

Physical excitations:

$$\begin{pmatrix} x^i & p^i \\ p^+ & x^- \end{pmatrix} \quad \begin{pmatrix} \alpha_n^i & \tilde{\alpha}_n^i \end{pmatrix}$$

center of mass
coords

transverse
oscillations

$x^+ \rightarrow$ can be absorbed
into τ

$\alpha_n^- \tilde{\alpha}_{-n}^- p^- \rightarrow$ satisfies (+)

Quantise

$$[x^i, p_j] = i \delta_j^i, \quad [x^-, p^+] = -i$$

$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = \hbar \delta^{ij} \delta_{n+m, n}$$

} recall cov.
quantisation

$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = \hbar \delta^{ij} \delta_{n+m,0} \quad \text{Quantisation}$$

Vacuum: $\hat{P}_\mu |0;p\rangle = p_\mu |0;p\rangle$ & $\tilde{\alpha}_n^i |0;p\rangle = \alpha_n^i |0;p\rangle = 0$
 $\forall n > 0$

Fock space: act w/ just transverse creation operators:
 α_{-n}^i & $\tilde{\alpha}_{-n}^i$

$\Rightarrow$ Manifestly unitary since norm is positive-definite

But, not all $\hat{P}_\mu$ are operators $\Rightarrow p_\mu$ not independent.

We should identify p^- of a state w/ the eigenvalue of the quadratic operator (+)

$\hookrightarrow$ We again have an ordering ambiguity in the quantum theory!

This ordering ambiguity affects the mass:

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n>0} \alpha_n^i \alpha_{-n}^i \right) \quad \text{classically}$$

$$\& M^2 = 2p^+ p^- - p^i p^i$$

$$\Rightarrow M^2 = \frac{4}{\alpha'} \sum_{n>0} \alpha_n^i \alpha_{-n}^i$$

In quantum theory, normal ordering constant a :

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \left(\frac{\alpha'}{2} p^i p^i + \sum_{n>0} \alpha_n^i \alpha_{-n}^i - a \right)$$

$$\Rightarrow M^2 = \frac{4}{\alpha'} \left(\sum_{n>0} \alpha_n^i \alpha_{-n}^i - a \right) = \frac{4}{\alpha'} \left(\sum_{n>0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i - a \right)$$

Define the level

$$N = \sum_{n>0} \alpha_n^i \alpha_{-n}^i, \quad M^2 = \frac{4}{\alpha'} (N - a) = \frac{4}{\alpha'} (\tilde{N} - a)$$

$$N_T = \sum_{i=1}^{D-2} \sum_{n \geq 0} \alpha_n^i \alpha_{-n}^i$$

$$\tilde{N}_T = \sum_{i=1}^{D-2} \sum_{n \geq 0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i$$

$$\Rightarrow M^2 = \frac{4}{\alpha'} (N_T - a) = \frac{4}{\alpha'} (\tilde{N}_T - a)$$

$$N_T = \tilde{N}_T \Rightarrow \text{level matching}$$

Normal ordering constant

Now let's fix a : The Lorentz symmetry is anomalous unless $a=1$ & $D=26$.

How to see this? We should check the Lorentz algebra:

$$[J^{\mu\nu}, J^{50}] \sim J$$

Messy $\rightarrow$ we'll use a little trick
& check whether the spectrum is Lorentz invariant.
(we'll eventually fix a & D properly using CFT)

Consider naive classical expression for the level

$$\frac{\alpha' p^-}{2} \supset \frac{1}{2} \sum_n \alpha_{-n}^i \alpha_n^i = \frac{1}{2} \sum_{n \leq 0} \alpha_{-n}^i \alpha_n^i + \frac{1}{2} \sum_{n \geq 0} \alpha_{-n}^i \alpha_n^i$$

& normal order:

$$= \frac{1}{2} \sum_{n \geq 0} \alpha_{-n}^i \alpha_n^i + \frac{1}{2} \sum_{n \leq 0} \alpha_n^i \alpha_{-n}^i - \frac{1}{2} \sum_{n \leq 0} [\alpha_n^i, \alpha_{-n}^i]$$

$$= \frac{1}{2} \sum_{n \geq 0} \alpha_{-n}^i \alpha_n^i + \frac{1}{2} \sum_{n \geq 0} \alpha_{-n}^i \alpha_n^i + \frac{1}{2} (D-2) \sum_{n \geq 0} n$$

$$= \frac{1}{2} \alpha_0^i \alpha_0^i + \sum_{n \geq 0} \alpha_{-n}^i \alpha_n^i + \frac{D-2}{2} \sum_{n \geq 0} n$$

this is zero-mode energy

this also appears in Hamiltonian of the string, since $H_{\text{c.c.}} \sim p^-$

(c.f. Exercise Sheet 4, Problem 3)

we will now show that

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$$\sum_{n>0} n = -\frac{1}{12}$$

by regularising:

$$\begin{aligned}\sum_{n=1}^{\infty} n &\rightarrow \sum_{n=1}^{\infty} n e^{-\epsilon n} \stackrel{\text{regulator}}{=} -\frac{\partial}{\partial \epsilon} \sum_{n=1}^{\infty} e^{-\epsilon n} \\ &= -\frac{\partial}{\partial \epsilon} (1 - e^{-\epsilon})^{-1} \\ &= \epsilon^{-2} - \frac{1}{12} + O(\epsilon)\end{aligned}$$

See exercise sheet 4, problem 5

The ϵ^{-2} term can be cancelled by adding cosm. constant counterterm in S_p .

or more formally, the ζ -function is defined as

$$\zeta(s) = \sum_{n \in \mathbb{Z}^+} n^{-s} \quad \text{for } \operatorname{Re}(s) > 0$$

and admits a unique analytic extension to all s , with

$$\zeta(-1) = -\frac{1}{12}.$$

$$-a = \frac{D-2}{2} \sum_{n>0} n \Rightarrow a = \frac{D-2}{24}$$

Mass formula:

$$M^2 = \frac{4}{\alpha'} \left(N_T - \frac{D-2}{24} \right) = \frac{4}{\alpha'} \left(\tilde{N}_T - \frac{D-2}{24} \right)$$

III - The closed string spectrum

We'll discuss the first few "levels" = eigenvalues of $N_T = \tilde{N}_T$.

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Level 0 - The tachyon

At level 0, we have no oscillator excitations

$\Rightarrow$ a unique state $|0;p\rangle$, the string vacuum

w/ $M^2 = -\frac{1}{\alpha'} \frac{D-2}{6}$ & space-time scalar $\Rightarrow$ no problem w/ Lorentz symm.

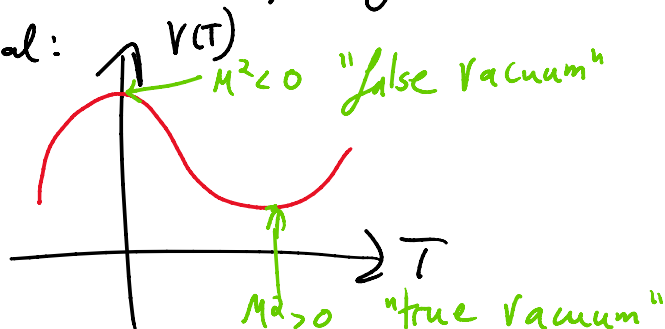
If $D > 2$, this is a tachyon: $M^2 < 0$

This signals an instability of the vacuum in QFT

Consider field $T(X) \rightarrow$ its mass is

$$M^2 = \frac{\partial^2 V(T)}{\partial T^2} \Big|_{T=0}$$

If $M^2 < 0$, we are expanding around a maximum of the potential:



Open question in bosonic string theory:

"Is there a true vacuum elsewhere?"

- The tachyon signals an instability of flat Mink. space w/ $T(X)=0$ & there is some true vacuum w/ some fields turned on

OR

- There is no stable vacuum & bosonic string theory is bad.

In the superstring, there is no tachyon.

Level 1 - Lorentz anomaly

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Level 1 - Lorentz anomaly

At level 1: $N_T = \tilde{N}_T = 1$

$$\Rightarrow \alpha_{-1}^i \tilde{\alpha}_{-1}^j |0; p\rangle$$

There are $(D-2)^2$ states w/

$$M^2 = \frac{4}{\alpha'} \left(1 - \frac{D-2}{24}\right)$$

Each oscillator transforms as a vector of $SO(D-2) \subset SO(1, D-1)$
which is manifest in light-cone gauge.

Can this be Lorentz invariant?

To get a clear answer, we go to rest frame
(Wigner classification) & there is a difference between
massive & massless particles

massive: rest frame $\rightarrow p^\mu = (p, 0, \dots, 0)$

$SO(D-1)$ symmetry "little group"

$\Rightarrow$ massive particles form reps of $SO(D-1)$

massless: rest frame $\rightarrow p^\mu = (p, 0, \dots, 0, p)$

$SO(D-2)$ symmetry "little group"

$\Rightarrow$ massless particles form reps of $SO(D-2)$

For example in $D=4$:

- massive particles labelled by $SO(3)$

$\Rightarrow$ spin $j \Rightarrow 2j+1$ states

- massless particles labelled by $SO(2)$

$\rightarrow$ helicity $\lambda \Rightarrow 1$ state

$\Rightarrow$ massive vector: 3 states

massless vector: 2 states (polarisations)

The fact that there are fewer states for massless particles
is related to gauge symmetry!

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At level 1, we have $(D-2) \times (D-2)$ states
decompose these into irreps of $SO(D-2)$:

symm traceless $\oplus$ antisymmetric $\oplus$ scalar (= trace)

↓ this must come from symm. traceless of $SO(D-1)$
but this would also give $SO(D-2)$ vector!

These cannot form $SO(D-1)$ reps!

$\Rightarrow$ Lorentz symmetry requires these to be massless:

$$M^2 = \frac{4}{\alpha'} \left(1 - \frac{D-2}{24} \right) = 0 \Rightarrow$$

$$\begin{aligned} D &= 26 \\ \alpha &= 1 \end{aligned}$$

Since these states are massless, we have gauge symmetry
& these particles are gauge bosons

$SO(24)$ symmetric traceless $\oplus$ antisymmetric $\oplus$ scalar

spacetime
fields

$G_{\mu\nu}$

Graviton

$B_{\mu\nu}$

Kalb-Ramond
(or 2-form)

ϕ

Dilaton

NB: Consistent quantisation of a massless spin-2 field requires
gauge symmetry to remove negative-norm states, c.f. AQ.
For massless spin-2 $\Rightarrow$ diffeomorphism required to decouple
ghosts

Include interactions preserving this $\Rightarrow$ GR

(c.f. Feynman lectures on Gravitation for nice account)

$\Rightarrow$ We can see that string theory contains GR.

Level 2

We have fixed $D=26$ & we now must hope that all
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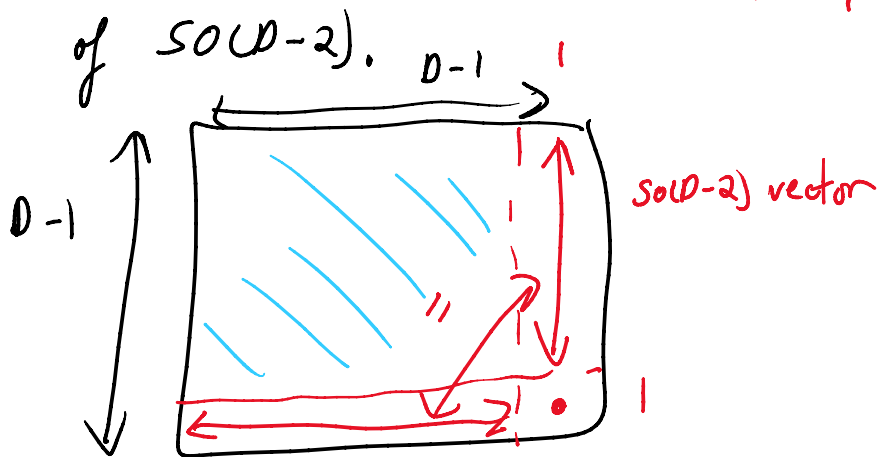
These will be massive $\Rightarrow$ form reps of $SO(D-1)$

Let's do $N_T = \tilde{N}_T = 2$ explicitly:

Can have $\alpha_{-1}^i \alpha_{-1}^j$ or α_{-2}^i in right-movers
& similar in left-movers

$$M^2 = \frac{4}{\alpha'} \alpha_1$$

The right-movers form $\underbrace{\text{symm. traceless} \oplus \text{scalar} \oplus \text{vector}}_{\alpha_{-1}^i \alpha_{-1}^j \quad \alpha_{-2}^i}$



$\Rightarrow$ These just form $SO(D-1)$ symm. traceless rep!

M_{ab} traceless: $M_{aa} = 0$

$a, b = 1, \dots, D-1$

$a = (i, z)$

$(1, \dots, D-2)$

$$\Rightarrow M_{11} + \underbrace{M_{zz}}_{\text{trace}} = 0$$

$(M_{ij} - \frac{1}{D-1} \delta_{ij} M_{kk})$ symm traceless of $SO(D-2)$

We have: $(\text{sym. traceless})_{D-1} \otimes (\text{sym. traceless})_{D-1} = (\text{irrep}_1 \oplus \text{irrep}_2 \oplus \dots)_{SO(D-1)}$

$\Rightarrow$ level 2 fits into $SO(D-1)$ rep for any D incl. $D=26$ ✓

Lorentz anomaly more properly

To see this holds for all levels & w/out using

$$\sum_{n \neq 0} n = -\frac{1}{12},$$

we should look at algebra of quantum operators comp. to the classical Lorentz generators.

E.g. the conformal symmetry is anomalous

since L_n & $\tilde{L}_n$ satisfy Virasoro algebra not Witt algebra (c.f. Exercise Sheet 3, Problem 3)

We should compute algebra of $J_{\mu\nu}$ (c.f. Sheet 3, p. 2) & require:

$$[J^{\mu\nu}, J^{\rho\sigma}] = 2\eta^{\mu\rho} J^{\sigma\nu} - 2\eta^{\nu\rho} J^{\sigma\mu}$$

The key computation:

$$[J^{i-}, J^{j-}] \stackrel{!}{=} 0$$

This involves p^-, α_n^- & $\tilde{\alpha}_n^-$ which are fixed by

$$\alpha_n^- = \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \sum_m \alpha_{n-m}^i \alpha_m^i + \sqrt{\frac{1}{2\alpha'}} \frac{1}{p^+} \delta_{n,0} a$$

& sim. for $\tilde{\alpha}_n^-$

$\Rightarrow$ One finds

$$[J^{i-}, J^{j-}] = \frac{2}{(p^+)^2} \sum_{n \neq 0} \left(\left[\frac{D-2}{24} - 1 \right] n + \left[a - \frac{D-2}{24} \right] \frac{1}{n} \right) (\alpha_n^i \alpha_n^j - \alpha_n^j \alpha_n^i) + (\alpha \leftrightarrow \tilde{\alpha})$$

$$= 0 \Leftrightarrow \boxed{D=26 \ \& \ a=1} \quad (\text{c.f. Weigand lecture notes})$$

We will be looking @ open strings

What boundary conditions are needed @ $\sigma=0, \pi$ to have a well-defined variational principle?