

Recap: Lightcone quantisation & spectrum

Lorentz invariance $\Leftrightarrow$ Level 1 $M^2 = 0 \Rightarrow D = 26$

Hamiltonian renormalisation $\rightarrow$ normal ordering constant

$$a = -\frac{D-2}{24}$$

Level 0 $\rightarrow$ tachyon: false vacuum

Level 2 $\rightarrow$ consistent^o with $D=26$

Level 1 $\rightarrow$ symmetric traceless $\oplus$ antisymmetric $\oplus$ scalar

Graviton

③ Open strings & D-branes

Highlights: • We saw closed string sector contains gravity amongst $m^2 = 0$ states

- Open string sector contains non-Abelian gauge bosons amongst $m^2 = 0$ states

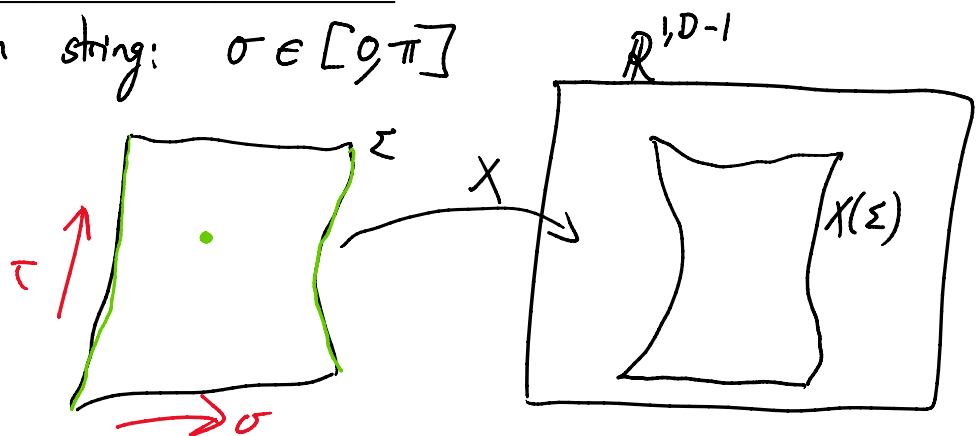
- D-branes $\rightarrow$ dynamical in string theory

- Geometric interpretation of Higgs mechanism

I - Open String Boundary conditions

Open string: $\sigma \in [0, \pi]$

Worksheet



Main difference to closed string: periodicity conditions
→ boundary conditions

I. e. @ any fixed τ , there are two distinguished $\sigma = 0, \pi$

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 $\Rightarrow$ Use the same action

$$S = \frac{-1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_{\alpha} X^{\mu} \partial_{\beta} X^{\nu} \eta_{\mu\nu}$$

$$= -\frac{1}{4\pi\alpha'} \int_0^{\pi} d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_{\alpha} X \cdot \partial_{\beta} X$$

In conformal gauge:

$$S = -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2\sigma \partial_{\alpha} X \cdot \partial^{\alpha} X$$

& supplement w/ B.C.'s s.t. variational principle works:

$$\delta S = -\frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \partial_{\alpha} \delta X \cdot \partial^{\alpha} X$$

$$= \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \delta X \cdot \partial_{\alpha} \partial^{\alpha} X$$

$$- \underbrace{\frac{1}{2\pi\alpha'} \int_{\partial\Sigma} n_{\alpha} \delta X^{\mu} \partial^{\alpha} X^{\nu} \eta_{\mu\nu}}_{\text{boundary term}}$$

usual: specify initial
 & final configurations
 $\delta X(\tau_i) = \delta X(\tau_f) = 0$

$$= \frac{1}{2\pi\alpha'} \int_{\Sigma} d^2\sigma \delta X \cdot \partial_{\alpha} \partial^{\alpha} X + \frac{1}{2\pi\alpha'} \left[\int_0^{\pi} d\sigma \dot{X} \cdot \delta X \right]_{\tau=\tau_i}^{\tau=\tau_f}$$

$$- \frac{1}{2\pi\alpha'} \left[\int_{\tau_i}^{\tau_f} d\tau X' \cdot \delta X \right]_{\sigma=0}^{\sigma=\pi}$$

Gives new BC's

We require: $\partial_{\sigma} X^{\mu} \delta X_{\mu} = 0$ @ $\sigma=0, \pi$

Two possibilities:

- Neumann boundary conditions (N)

$$\partial_{\sigma} X^{\mu} = 0 \quad @ \quad \sigma=0, \pi$$

- Dirichlet boundary conditions (D)

$$\delta X^{\mu} = 0 \quad @ \quad \sigma=0, \pi$$

... ..

$$\delta X^M = 0 \quad @ \sigma = 0, \pi$$

We can have different BC's @ $\sigma = 0, \pi$ & different BC's for each space-time direction

$\Rightarrow$ 4 possibilities for each direction

$$\sigma = 0, \pi$$

$$N - N$$

$$D - D$$

$$N - D$$

$$D - N$$

$\} \rightarrow$ cf. Exercise Sheet 4, Problem 4

What do these BC's mean?

- N: $X^M = 0, \delta X^M \neq 0$ @ $\sigma = 0, \pi \Rightarrow$ end of string can move freely but has no slope



In static gauge $X^0 \equiv t = R \tau$

$$\text{Constraints: } \dot{\underline{x}} \cdot \underline{x}' = 0 \text{ \& } \underline{\dot{x}}^2 + \underline{x}'^2 = R^2$$

$$\Rightarrow \left| \frac{d\underline{x}}{d\tau} \right| = 1 \text{ @ endpoint } \Rightarrow \text{moves @ speed of light}$$

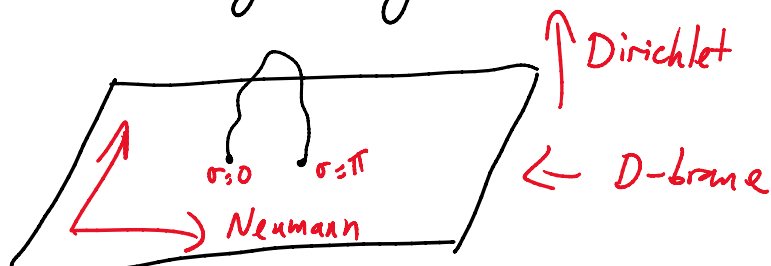
- D: $\delta X^M = 0 \& X'^M \neq 0$ @ $\sigma = 0, \pi \Rightarrow$ endpoint is fixed!

$$X^M(\sigma=0) = C^M = \text{constant}$$

Why should endpoint be fixed?

Breaks translation invariance $\Rightarrow$ long ignored

Modern understanding: string ends on a D-brane



Consider N-N BC's in some directions $\partial_a X^a = 0, a = 0, \dots, p$

Consider N-N BC's in some directions $\partial_\sigma X^a = 0, a = 0, \dots, p$
D-D BC's in remaining directions $X^I|_{\sigma=0,\pi} = c^I, I = p+1, \dots, D-1$

$\Rightarrow$ End-points lie on $(p+1)$ -dim hypersurface D_p -Brane
BC's break Lorentz group $SO(1, D-1) \rightarrow SO(1, p) \times SO(D-p-1)$

E.g. $D0 \rightarrow$ particle

$D1 \rightarrow$ string (D-string)

$D2 \rightarrow$ membrane

etc.

$D(-1) \rightarrow$ instanton (string localised in time)

$D(D-1) \rightarrow$ spacetime-filling D-brane $\Leftrightarrow$ NN conditions everywhere!

II - Open string mode expansion

N-N BC's $\partial_\sigma X^a, a = 0, \dots, p$

D-D BC's $X^I|_{\sigma=0,\pi} = c^I, I = p+1, \dots, D-1$

EOM $\partial_\alpha \partial^\alpha X = 0$ as before $\Rightarrow$ left/right X 's:

$$X = X_L(\sigma^+) + X_R(\sigma^-)$$

$$X_L^M(\sigma^+) = \frac{1}{2} (x^M + y^M) + \alpha' p_L^M \sigma^+ + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n^M e^{-in\sigma^+}$$

$$X_R^M(\sigma^-) = \frac{1}{2} (x^M - y^M) + \alpha' p_R^M \sigma^- + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n^M e^{-in\sigma^-}$$

factor of 2 different from closed string

$$P^M = \frac{1}{2\pi\alpha'} \int_0^\pi d\sigma \dot{X}^M = p^M = (p_L^M + p_R^M)$$

BC's: (i) $\partial_\sigma X^a = 0$ @ $\sigma = 0, \pi$

$$\Rightarrow p_L^a = p_R^a \text{ \& \; } \tilde{\alpha}_n^a = \alpha_n^a$$

(ii) $X^I = c^I$ @ $\sigma = 0, \pi$

$$\Rightarrow x^I = c^I, p_L^I = p_R^I = 0, \tilde{\alpha}_n^I = -\alpha_n^I$$

In both cases, only one set of oscillators as BC's relate α_n & $\tilde{\alpha}_n$

Endpoints on diff. branes

$$X^I \sim c^I + \frac{\alpha^I \cdot c^I}{\pi} \sigma + \text{osc.}$$

In both cases, only one set of oscillators relate α_n & $\tilde{\alpha}_n$

III - Quantisation & spectrum

Dynamical fields are x^a, p^a, α_n^M , rest fixed by B.C.'s

$\hookrightarrow$ c.o.m. on the brane only

$\Rightarrow$ In quantum theory, states are localised on the D-brane!

... $\alpha_{-1}^M |0; x^a\rangle$

For spectrum, easiest light-cone quantisation:

$$X^\pm = \frac{1}{\sqrt{2}} (X^0 \pm X^p)$$

$\nwarrow \nearrow$
light-cone
on brane

Very similar as for closed strings but w/ only one set of oscillators α_n^i

$$M^2 = \frac{1}{\alpha'} \left(\sum_{i=1}^{p-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{i=p+1}^{D-1} \sum_{n>0} \alpha_{-n}^i \alpha_n^i - a \right)$$

$\nearrow$
factor of 4
rel. to closed
string due to
 $p \rightarrow 2p$

$\nwarrow$
normal ordering
constant

$$a = - \frac{D-2}{12} \sum_{n>0} n$$

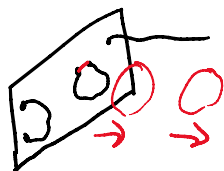
$$= \frac{D-2}{24}$$

We find as for closed string

$D=26, a=1$

in order not to have an anomaly in $SO(1,p) \times SO(D-1-p) \subset SO(D-1)$

NB: open & closed strings are different states in same theory





Open strings can join to form closed strings
 Most superstring theories include both open & closed strings.
 (Heterotic string theory)

Vacuum

As usual: $|0;p\rangle$ s.t. $\alpha_n^i |0;p\rangle = 0 \quad \forall n > 0$

& $i = 1, \dots, p-1, p+1, \dots, D-1$

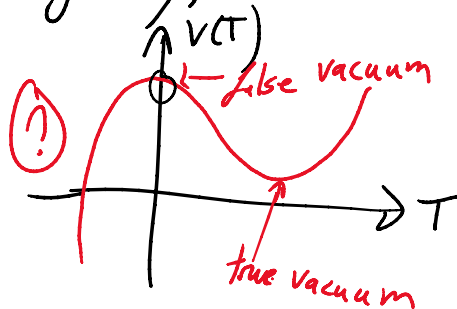
& $\hat{p}^a |0;p\rangle = p^a |0;p\rangle$

(p^- fixed by mass-shell condition)

$$M^2 = -\frac{1}{\alpha'} \quad \text{tachyonic}$$

Just like closed string theory $\rightarrow$ instability of vacuum.

In bosonic open string theory, a "true vacuum" is known



In superstring theory, this is absent!

Level 1 - photons

At level 1, we can have either

$$\begin{array}{l}
 \left. \begin{array}{l} \text{vector of } so(p-1) \\ \downarrow \\ so(p) \text{ } A_a \text{ gauge boson} \Rightarrow \text{spin-1 on brane} \end{array} \right\} \left. \begin{array}{l} \text{massless} \\ \text{little group} \\ so(p-1) \end{array} \right\} \alpha_{-1}^{\bar{a}} |0;p\rangle \quad \bar{a} = 1, \dots, p-1 \\
 \left. \begin{array}{l} \text{scalars of } so(1,p) \\ \Rightarrow \text{scalars on brane} \end{array} \right\} \alpha_{-1}^I |0;p\rangle \quad I = p+1, \dots, D-1
 \end{array}
 \quad \left. \begin{array}{l} \text{OR} \\ \end{array} \right\} M^2 = 0$$

$SO(1,p)$
 $\Rightarrow$ scalars on brane
 & vector of $SO(D-1-p)$

These need to be massless in order to be compatible with $SO(1,p) \times SO(D-1-p)$ by same arguments as for closed string *cf. Exercise Sheet 5*

We find a massless vector gauge boson A_a & scalar ϕ^I transform under global $SO(D-1-p)$ rotations.

The natural interpretation of ϕ^I is as fluctuations of the D_p -brane



(i) This is the first sign that D-branes are dynamical objects.

(ii) Also, D-D string excitations carry momentum $\perp$ Dp-branes *cf. Exercise Sheet 5*

$$T_p \sim \frac{1}{\ell_{\text{string}}} \Rightarrow \text{very heavy in perturbation theory}$$

(iii) String theory is gravitational theory so shouldn't rigid objects

Level 2 *cf. Exercise Sheet 5*

IV - Non-Abelian gauge groups

So far, we've only considered 1 D-brane. Now, let's consider multiple ones





$$X^I(0, \tau) = c^I \text{ \& } X^I(\pi, \tau) = d^I$$

Mode expansion (c.f. Exercise Sheet 4 Qu. 4)

$$X^I = c^I + \frac{(d^I - c^I)\sigma}{\pi} + \text{oscillators}$$

$$\partial_+ X \cdot \partial_+ X = \alpha' p^2 + \frac{|d - c|^2}{4\pi^2} + \text{osc.} = 0$$

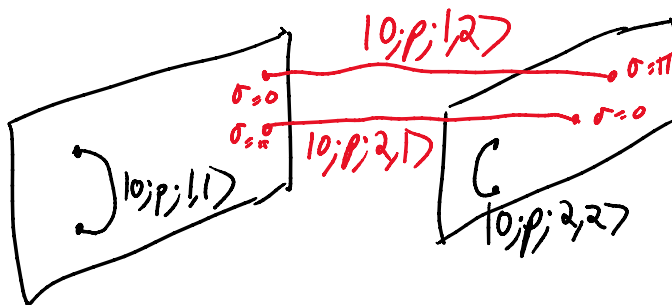
$\Rightarrow$ Mass-shell condition is modified to

$$M^2 = \frac{|d - c|^2}{(2\pi\alpha')^2} + \frac{1}{\alpha'} (N - 1)$$

↑ level ↑ normal ordering constant

Masses are shifted } by potential energy of stretched string $\Delta E = \tau |d - c|$

The strings can now be distinguished by which brane they start/end on, e.g.



& similarly with excitations: $\alpha_{-1}^9 |10; p; 1, 1> \rightarrow A_{\mu}$ massless vector

$\alpha_{-1}^9 |10; p; 1, 2> \rightarrow A_{\mu}$ massive vector

massless scalars & massive scalars

As we bring branes together $d^I \rightarrow c^I$, the massive vectors & scalars become massless $\rightarrow$ 4 massless vectors

As we bring branes together $d \rightarrow c\mathbb{I}$, the massive vectors & scalars become massless $\rightarrow$ 4 massless vectors & 4 massless scalars

We should also consider N coincident D-branes & include Chan-Paton factors labelling the start & endpoints of strings:

e.g. @ level 1 N^2 massless vectors $\alpha_{-1}^{\mu} |0; p; m, n\rangle$
 N^2 massless scalars $\alpha_{-1}^I |0; p; m, n\rangle$

let's introduce a basis: λ_m^A $m, n = 1, \dots, N$
 $A = 1, \dots, N^2$

$$|0; p; m, n\rangle = \lambda_m^A |0; p; A\rangle$$

we can make this Hermitian $(\lambda^A)^{\dagger} = \lambda^A$

$\Rightarrow$ We get N^2 Hermitian gauge fields A_{μ}^A
 & N^2 Hermitian scalar fields ϕ^I

As we'll see later, gauge group $\rightarrow U(N)$ non-Abelian
 w) $\phi^I \in \text{adj}(U(N))$

Moving coincident D-branes apart gives a geometric realisation of the Higgs effect (with scalar fields in adjoint representation):

Strings stretched between D-branes give massive vector & massive scalar fields associated w/ symmetry breaking

We see that string theory contains gravity in closed string sector & non-Abelian gauge theories in open string sector coming from coincident D-branes

string sector & non-abelian gauge theories in open
string sector coming from coincident D-branes

- geometric understanding of gauge theories
- AdS/CFT
- other gauge groups possible via orientifolds
c.f. Exercise Sheet 5

Which rotate & use $\mathbb{C}$ -coords

$$(\tau, \sigma) \rightarrow (\sigma^1, \sigma^2) = (\sigma, i\tau)$$

$$z = \sigma^1 + i\sigma^2$$

Flat metric $ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 \rightarrow dz \text{ \& \& } d\bar{z}$?

Consider ∂_z & $\partial_{\bar{z}}$ & $\square X = 0$ in terms of ∂_z & $\partial_{\bar{z}}$

Tracelessness of $T^\alpha_\alpha = 0 \Rightarrow T_{zz}, T_{\bar{z}\bar{z}}, T_{z\bar{z}}$