

Recap:

- Open string & dynamics $\rightarrow$ still Polyakov
- 2 BC's: Dirichlet & Neumann
- D-branes: ^{red arrow} dynamical in quantum theory
- Mass spectrum $\rightarrow$ 0th level: tachyon, but true vacuum exists
1st level: gauge bosons
scalar fields (fluctuations of D-brane)
- Single D-brane: $U(1)$ gauge theory
- Multiple D-brane: $U(N)$ gauge theory for N coincident branes
- Moving branes apart $\Leftrightarrow$ Higgs effect (fields becoming massive)

④ Conformal Field Theory

The next few lectures will be very abstract & technical.
But it will come together eventually & help us.

We want to study theories that are invariant under conformal transformations:

$$\begin{aligned} \sigma^\alpha &\rightarrow \tilde{\sigma}^\alpha(\sigma) \\ \text{st. as a diffeo, we would have} \\ g_{\alpha\beta}(\sigma) &\rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \Omega^2(\sigma) g_{\alpha\beta}(\sigma) \end{aligned} \quad \left. \vphantom{\begin{aligned} \sigma^\alpha &\rightarrow \tilde{\sigma}^\alpha(\sigma) \\ g_{\alpha\beta}(\sigma) &\rightarrow \tilde{g}_{\alpha\beta}(\tilde{\sigma}) = \Omega^2(\sigma) g_{\alpha\beta}(\sigma) \end{aligned}} \right\} (*)$$

If g is dynamical, this is just a special diffeo
 $\Rightarrow$ a gauge symmetry & not what we want.

We want to study theories w/ fixed background metric
invariant under conformal transformations of the type (*)

conformal transformations of the type (*)

Equivalently, *(classically)* the theory can be coupled to a dynamical metric & is invariant under

$$\text{Weyl transformations: } g_{\alpha\beta} \rightarrow \Omega^2(\sigma) g_{\alpha\beta}$$

These theories are invariant under scaling transformations.

E.g. Free scalar field in 2-D

$$S_p = -\frac{1}{2} \int d^2\sigma \sqrt{g} (g^{\alpha\beta} \partial_\alpha X \partial_\beta X - \cancel{m^2 X^2})$$

Not conformal in QFT $\rightarrow$ Maxwell theory in 4-D

Conformal in QFT $\rightarrow$ YM theory in 4-D

I - Complex coordinates

To make life easier, we will go to Euclidean signature w/

$$(\tau, \sigma) \rightarrow (\sigma^1, \sigma^2) = (\sigma, i\tau) \quad \left. \vphantom{(\sigma^1, \sigma^2)} \right\} \text{ Wick rotation}$$

The light-coordinates, that were useful in studying conf. transformations,

$$\sigma^\pm = \tau \pm \sigma$$

now become

$$Z = \sigma^1 + i\sigma^2$$

$$\& \bar{Z} = \sigma^1 - i\sigma^2$$

Holomorphic $\Leftrightarrow$ "left-moving"

Anti-Holo. $\Leftrightarrow$ "right-moving"

" " " " " " " " " " " "

Anti-Holo. $\leftrightarrow$ "right-moving"

Holo. derivatives: $\partial = \partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$

$$\bar{\partial} = \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$$

Flat metric: $ds^2 = (d\sigma^1)^2 + (d\sigma^2)^2 = dz d\bar{z}$



$$g_{zz} = g_{\bar{z}\bar{z}} = 0$$

$$g_{z\bar{z}} = \frac{1}{2}$$

$\Rightarrow$ Measure: $dz d\bar{z} = 2 d\sigma^1 d\sigma^2$

& care needs to be taken w/ δ -fcts:

$$\int d^2z \delta(z, \bar{z}) = 1 \text{ c.f. } \int d^2\sigma \delta(\sigma) = 1$$


relative
factor
of
2

$$V^z = V^1 + iV^2, V^{\bar{z}} = V^1 - iV^2$$

$$V_{\bar{z}} = \frac{1}{2}V^z \quad \& \quad V_z = \frac{1}{2}V^{\bar{z}}$$

Wave equation: $\square X = 2\partial\bar{\partial}X = 4\partial\bar{\partial}X = 0$

$$\Rightarrow \underline{X(z, \bar{z})} = \underline{X(z)} + \underline{\bar{X}(\bar{z})}$$

Like in light-cone coords, Conf. transformations are
holomorphic transformations:

$$\begin{aligned} z &\rightarrow z' = f(z) \\ \bar{z} &\rightarrow \bar{z}' = \bar{f}(\bar{z}) \end{aligned} \quad \text{w/} \quad ds^2 = dz d\bar{z} \rightarrow ds^2 = \left| \frac{df}{dz} \right|^2 dz d\bar{z}$$

$$\bar{z} \rightarrow \bar{z}' = f(\bar{z})$$

Throughout, we will treat z & $\bar{z}$ as independent but at end restrict to $\bar{z} = z^*$.

II - Stress-energy tensor

Recall, from Exercise Sheet 2, for any conformal theory

$$T^\alpha_\alpha = 0$$

$$\Rightarrow T_{z\bar{z}} = 0 \quad \Rightarrow T_{\alpha\beta} = (T_{zz}, T_{\bar{z}\bar{z}})$$

$$\text{conservation law: } \partial_\alpha T^{\alpha\beta} = 0$$

$$\partial_z T^{z\bar{z}} = 0 \quad \& \quad \partial_{\bar{z}} T^{\bar{z}z} = 0$$

$$\Rightarrow \partial T_{z\bar{z}} = 0 \quad \& \quad \bar{\partial} T_{\bar{z}z} = 0$$

$$\Rightarrow T_{zz} = T(z), \quad T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$$

As we saw, conformal currents

$$J^\alpha_E = T^{\alpha\beta} E_\beta$$

$$\forall \text{ CKV } E: \underbrace{p \cdot E = 0}_{0 = \mathbb{R} \epsilon_\beta + \mathbb{R} \bar{\epsilon}_\beta - \frac{1}{2} g_{\alpha\beta} \bar{\epsilon}_\alpha \epsilon^\beta} \Leftrightarrow E^\alpha = \begin{pmatrix} E^z \\ E^{\bar{z}} \end{pmatrix}$$

$$w) \quad E_{\bar{z}} = E_{\bar{z}}(z) = E(z)$$

$$E_z = E_z(\bar{z}) = \bar{E}(\bar{z})$$

$$J^z_E = 0 \quad \& \quad J^{\bar{z}}_E = 2T(z) E(z)$$

$$J^{\hat{z}}_{\bar{E}} = 0 \quad \& \quad J^z_{\bar{E}} = 2\bar{T}(\bar{z}) \bar{E}(\bar{z})$$

$$\text{unsolved: } \partial_\alpha J^\alpha = \partial J^z + \bar{\partial} J^{\bar{z}} = 0$$

Unsoved: $\partial_\alpha \mathcal{J}^\alpha = \underbrace{\partial_z^2}_{=0} + \underbrace{\bar{\partial}^2}_{=0} \mathcal{J}^\alpha = 0$

E.g. free scalar fields

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_\alpha X \partial^\alpha X = \frac{1}{2\pi\alpha'} \int d^2z \partial X \bar{\partial} X$$

$$T_{\alpha\beta} = -\frac{1}{\alpha'} \left(\partial_\alpha X \partial_\beta X - \frac{1}{2} \delta_{\alpha\beta} (\partial X)^2 \right) \quad \left| \delta_{z\bar{z}} = \frac{1}{2} \right.$$

$$\Rightarrow T_{z\bar{z}} = 0 \quad \& \quad T = -\frac{1}{\alpha'} \partial X \partial X$$

$$\bar{T} = -\frac{1}{\alpha'} \bar{\partial} X \bar{\partial} X$$

$$\text{EOM: } \partial \bar{\partial} X = 0 \Rightarrow X(z, \bar{z}) = X(z) + \bar{X}(\bar{z})$$

$\Rightarrow T \& \bar{T}$ become holomorphic & anti-holomorphic
 $\Rightarrow$ conserved

III - Quantum CFT

CFT use slightly different language to QFT:		
	<u>QFT</u>	<u>CFT</u>
"field"	Dynamical DoF e.g. in action, X	All <u>local operators</u> : $X, \partial X, \partial^2 X, \dots, \partial^n X$ $X^2, X^3, \dots, e^{iX}$

(i) Operator Product Expansion

OPE: The product of any two local operators at nearby points can be approximated by a string of operators at one of the two points

$$O_i(z, \bar{z}) O_j(w, \bar{w}) = \sum_k C_{ij}^k(z-w, \bar{z}-\bar{w}) O_k(w, \bar{w})$$

e.g. $X(z, \bar{z}) X(w, \bar{w}) = \left[X(w, \bar{w}) + (z-w) \partial X(w, \bar{w}) + (\bar{z}-\bar{w}) \bar{\partial} X(w, \bar{w}) + |z-w|^2 \partial \bar{\partial} X(w, \bar{w}) + \dots \right] X(w, \bar{w})$

Here $C_{XX}^{X^2} = 1$, $C_{XX}^{\partial X} = (z-w)$, $C_{XX}^{\bar{\partial} X} = (\bar{z}-\bar{w})$,
 $C_{XX}^{\partial \bar{\partial} X} = |z-w|^2, \dots$

Translational invariance C_{ij}^k depend only on $(z-w)$
 & $(\bar{z}-\bar{w})$

We are working in quantum theory, so we will always take the OPE to mean as evaluated w/ generic operator insertions inside time-ordered correlation functions:

$$\langle O_i(z, \bar{z}) O_j(w, \bar{w}) \dots \rangle = \sum_k C_{ij}^k(z-w, \bar{z}-\bar{w}) \langle O_k(w, \bar{w}) \dots \rangle$$

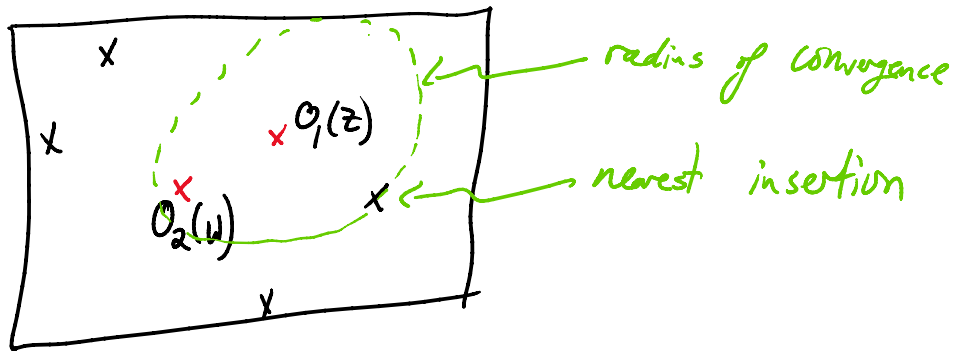
NB: • Correlation functions are time-ordered $\Rightarrow$ operators commute:

$$O_i(z, \bar{z}) O_j(w, \bar{w}) = O_j(w, \bar{w}) O_i(z, \bar{z})$$

w/ minus sign for Grassmann-valued operators (later!)

- "Generic operator insertions" means arbitrary operator insertions at distance large compared to $|z-w|$.
- In general QFT, OPE are approximate w/ finite radius of convergence depending on O_i, O_j . In CFT, OPEs are exact w/ radius of convergence given by the nearest operator insertion

the nearest operator insertion



- In general, OPEs are singular as $z \rightarrow w$.
This is actually the physically interesting part!

commutation relations
symmetry transformations

We will often denote the finite part as ...

- We will be working w/ path integrals s.t.

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

measure $\rightarrow$ collection of all fields (QFT sense!) $\leftarrow$ action

Time-ordered correlation functions:

$$\langle O; O_j \dots \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, e^{-S[\phi]} O_i O_j \dots$$

(ii) Symmetries

Classical field theory: Noether's Thm $\Rightarrow$ Conserved quantities
Real: requires EOM!

QFT : $\Rightarrow$ Ward identities

QFT : $\Rightarrow$ Ward identities

Symmetry in QFT: $\phi \rightarrow \phi' = \phi + \epsilon \delta\phi$
s.t. action & measure are invariant

[strictly, only the product needs to be invariant]

classical symmetry $\rightarrow S[\phi'] = S[\phi]$ & $D\phi' = D\phi$ $\leftarrow$ Quantum requirement: if this does not hold, the symmetry is anomalous
 $\Rightarrow Z = \int D\phi e^{-S[\phi]}$ is invariant

let's use the same trick as in Ex. sheet 1, Qu. 3

take global symmetry & promote $\epsilon \rightarrow \epsilon(x)$

$\Rightarrow$ change must be proportional to $\partial\epsilon$:

$$\begin{aligned} Z &\rightarrow \int D\phi' e^{-S[\phi']} \\ &= \int D\phi e^{-S[\phi] - \frac{1}{2\pi} \int \mathcal{J}^\alpha \partial_\alpha \epsilon} \\ &= \int D\phi e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int \mathcal{J}^\alpha \partial_\alpha \epsilon \right) \\ &= Z \quad \text{since } \phi' \rightarrow \phi \text{ is just a relabelling of a dummy variable} \end{aligned}$$

$\frac{1}{2\pi}$ convention
shorthand $\int d^2x \sqrt{g}$
includes possible contributions from measure $D\phi'$

$$\Rightarrow \int D\phi e^{-S[\phi]} \left(\int \mathcal{J}^\alpha \partial_\alpha \epsilon \right) = 0 \quad \forall \epsilon$$

$$\langle \partial_\alpha \mathcal{J}^\alpha \rangle = 0 \quad (\text{like Ehrenfest})$$

Now, consider a correlation fct w/ insertions

Now, consider a correlation for w/ insertions

$$\langle O_1(\sigma_1) O_2(\sigma_2) \dots O_n(\sigma_n) \rangle = \frac{1}{Z} \int D\phi e^{-S[\phi]} O_1(\sigma_1) \dots O_n(\sigma_n)$$

Recall O_i are made up of the " ϕ fields"

$$\Rightarrow \text{under } \phi \rightarrow \phi' = \phi + \epsilon \delta\phi,$$

$$O_i \rightarrow O_i + \epsilon \delta O_i$$

Same trick: $\epsilon \rightarrow \epsilon(\sigma)$

$$\Rightarrow \frac{1}{Z} \int D\phi e^{-S[\phi]} \left(1 - \frac{1}{2\pi} \int \mathcal{J}^\alpha \partial_\alpha \epsilon \right) (O_1 + \underbrace{\epsilon(\sigma_1) \delta O_1(\sigma_1)}_{\text{red}}) O_2 \dots O_n + \dots$$

$$= \frac{1}{Z} \int D\phi e^{-S[\phi]} O_1 \dots O_n$$

Infinitesimally:

$$- \frac{1}{2\pi} \int \epsilon(\sigma) \partial_\alpha \langle \mathcal{J}^\alpha O_1 \dots O_n \rangle = \epsilon(\sigma_1) \langle \delta O_1 O_2 \dots O_n \rangle + \epsilon(\sigma_2) \langle O_1 \delta O_2 \dots O_n \rangle + \dots$$

$$= \int \left[\delta(\sigma - \sigma_1) \epsilon(\sigma) \langle \delta O_1 O_2 \dots O_n \rangle + \delta(\sigma - \sigma_2) \epsilon(\sigma) \langle O_1 \delta O_2 \dots O_n \rangle + \dots \right]$$

$\Rightarrow$ Ward identity

other insertions @ $\sigma_n \neq \sigma, \sigma_1$

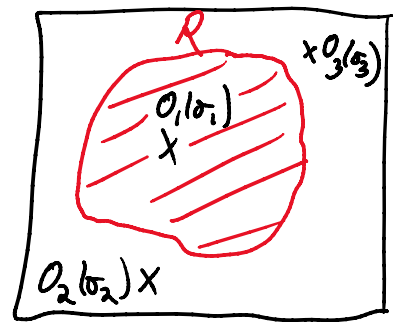
$$- \frac{1}{2\pi} \partial_\alpha \langle \mathcal{J}^\alpha O_1(\sigma_1) \rangle = \delta(\sigma - \sigma_1) \delta O_1$$

As an integral:

$$- \frac{1}{2\pi} \partial_\alpha \langle \mathcal{J}^\alpha O_1(\sigma_1) \rangle = \delta O_1(\sigma_1)$$

$$-\frac{1}{2\pi} \int_R \partial_\alpha (J^\alpha O_1(\sigma_1)) = \oint O_1(\sigma_1)$$

← integral over a region including only σ_1



In 2-d, we can use Stoke's Thm to write

$$\begin{aligned} -\frac{1}{2\pi} \int_R \partial_\alpha J^\alpha &= -\frac{1}{2\pi} \oint_{\partial R} n_\alpha J^\alpha = -\frac{1}{2\pi} \oint_{\partial R} (J_1 d\sigma^2 - J_2 d\sigma^1) \\ &= \frac{i}{2\pi} \oint_{\partial R} (J_z dz - J_{\bar{z}} d\bar{z}) \end{aligned}$$

$$\Rightarrow \frac{i}{2\pi} \oint_{\partial R} dz \langle \underline{J_z(z, \bar{z})} O_1(\sigma_1) \dots \rangle$$

$$- \frac{i}{2\pi} \oint_{\partial R} d\bar{z} \langle \underline{J_{\bar{z}}(z, \bar{z})} O_1(\sigma_1) \dots \rangle = \langle \oint O_1(\sigma_1) \dots \rangle$$