Lecture 8

Wednesday, 9 December 2020 09:12

Recap:

- CFT & Wick rotated → $\mathbb{C}$ coords
- Holomorphicity ⟺ left/right moving
- Holomorphicity of conformal transformations
- In 2d CFT, $T_{z\bar{z}} = 0$, & $T_{zz} = T(z)$ & $T_{\bar{z}\bar{z}}(\bar{z})$
- OPE's: expansion of nearby operators in correlation fct's (time-ordered),
  exact & radius of convergence given by nearest insertion
- Symmetries in QFT ⟺ Ward identities,
  quantum analogue of Noether's Thm
  $\langle \partial_\alpha J^\alpha \rangle = 0$

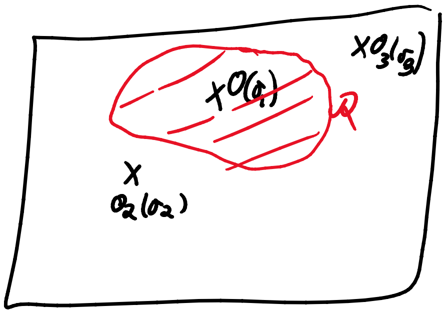
Last time, we met Ward identities:

$$-\frac{1}{2\pi}\partial_\alpha\left(J^\alpha O_1(\sigma_1)\right) = \delta(\sigma-\sigma_1)\,\delta O_1$$

other insertions are @ $\sigma_n \neq \sigma, \sigma_1$

which when integrating over region containing only σ_1 gives:

$$-\frac{1}{2\pi}\int_R \underset{=}{\partial_\alpha}\left(J^\alpha O_1(\sigma_1)\right) = \delta O_1(\sigma_1)$$



$$\Rightarrow \frac{i}{2\pi}\oint_{\partial R} dz \,\langle J_z(z,\bar{z})\, O_1(\sigma_1)\ldots\rangle$$

$$-\frac{i}{2\pi}\oint_{\partial R} d\bar{z}\,\langle J_{\bar{z}}(z,\bar{z}) O_1(\sigma_1)\ldots\rangle$$

$$= \langle \delta O_1(\sigma_1)\ldots\rangle$$

Ward identities for conformal transformations

Word identities for conformal transformations

For conf. transf. $J_z(\bar z)$ holomorphic & $J_{\bar z}(\bar z)$ anti-holomorphic

$$\frac{i}{2\pi}\oint_{\partial R} dz\ \underbrace{J_z(z)\,O_1(\sigma_1)}_{\text{Expand as OPE}} = \underbrace{-\,\mathrm{Res}\left[J_z\,O_1\right]}$$

$$J_z(z)\,O_1(w,\bar w) = \ldots + \frac{\mathrm{Res}\left[J_z\,O_1(w,\bar w)\right]}{z-w} + \ldots$$

For conformal transformations $\delta z = \epsilon(z)$, $\delta\bar z = \bar\epsilon(\bar z)$,
currents are $J_z = T(z)\,\epsilon(z)$ & $J_{\bar z} = \bar T(\bar z)\,\bar\epsilon(\bar z)$

independent

$$\Rightarrow\ \delta O_1(\sigma_1) = -\,\mathrm{Res}\left[\epsilon(z)\,T(z)\,O_1(\sigma_1)\right] \quad \text{under } \delta z = \epsilon(z)$$

$$\delta O_1(\sigma_1) = -\,\mathrm{Res}\left[\bar\epsilon(\bar z)\,\bar T(\bar z)\,O_1(\sigma_1)\right] \quad \text{under } \delta\bar z = \bar\epsilon(\bar z)$$

We see that OPE of O & $T, \bar T \Rightarrow$ transformation rule under conformal transformations

Conversely, conformation transformation rules of O $\Rightarrow$ one singular part of OPE of O & $T, \bar T$!

(iii) Primary Operators

Let's start putting together pieces of OPEs from transf. properties.

2 simple kinds of conformal transformations:

- translations $\delta z = \epsilon(z) = \epsilon$ = constant
- scaling & rotations $\delta z = \epsilon(z) = \epsilon \times z$ $\{\epsilon$ = real: scaling

– Scaling & rotations $\delta z = \epsilon(z) = \epsilon \times z$ $\begin{cases} \epsilon = \text{real: scaling} \\ \epsilon = \text{imaginary: rotation} \end{cases}$

Translations for all operators

$$\Rightarrow \mathcal{O}(z-\epsilon, \bar{z}-\bar{\epsilon}) = \mathcal{O}(z,\bar{z}) - \epsilon\,\partial\mathcal{O}(z,\bar{z}) - \bar{\epsilon}\,\bar{\partial}\mathcal{O}(z,\bar{z}) + \dots$$

Word identity implies:
$$\partial\mathcal{O}(w,\bar{w}) = \mathrm{Res}\left[T(z)\,\mathcal{O}(w,\bar{w})\right]$$
$$\bar{\partial}\mathcal{O}(w,\bar{w}) = \mathrm{Res}\left[\bar{T}(\bar{z})\,\mathcal{O}(w,\bar{w})\right]$$

$\Rightarrow$ All operators must have OPE:

$$T(z)\,\mathcal{O}(w,\bar{w}) = \dots + \frac{\partial\mathcal{O}(w,\bar{w})}{z-w} + \dots$$

$$\bar{T}(\bar{z})\,\mathcal{O}(w,\bar{w}) = \dots + \frac{\bar{\partial}\mathcal{O}(w,\bar{w})}{\bar{z}-\bar{w}} + \dots$$

Rotations & scalings

Def^n: An operator has weight $(h,\tilde{h})$ if under $\delta z = \epsilon z$, $\delta\bar{z} = \bar{\epsilon}\bar{z}$, $\mathcal{O}$ transforms as

$$\delta\mathcal{O}(z,\bar{z}) = -\epsilon(h\,\mathcal{O} + z\,\partial\mathcal{O}) - \bar{\epsilon}(\tilde{h}\,\mathcal{O} + \bar{z}\,\bar{\partial}\mathcal{O})$$

come from Taylor expanding $\mathcal{O}(z-\epsilon z, \bar{z}-\bar{\epsilon}\bar{z})$

— terms involving $(h,\tilde{h})$ are special & only for eigenstates of dilatations & rotations.

NB:
- h & $\tilde{h}$ are real & we'll later see unitarity requires $h,\tilde{h} \geq 0$.
- Eigenvalue under dilatations: $\epsilon = \bar{\epsilon} = \text{real}$

  $\Delta = h + \tilde{h} \equiv$ scaling dimension
- Eigenvalue under rotations: $\epsilon = -\bar{\epsilon} = \text{imaginary}$

- Eigenvalue under rotations: $\epsilon = -\bar{\epsilon}$ = imaginary

$$s = h - \tilde{h} \equiv \text{spin}$$

- Scaling dimension is "dimension" in dimensional analysis

$$\text{e.g. } \Delta[\partial] = +1$$

Quantum effects can change classical scaling dimension!
→ "Anomalous dimensions"

Let's look at OPE of operator O of weight $(h, \hat{h})$:

current is
$$J_z = T(z)\,\epsilon(z) = \epsilon\, T(z)\, z$$
$$J_{\bar{z}} = \bar{T}(\bar{z})\,\bar{\epsilon}(\bar{z}) = \bar{\epsilon}\, \bar{T}(\bar{z})\, \bar{z}$$

$$T(z)\,O(w,\bar{w}) = \cdots + h\frac{O(w,\bar{w})}{(z-w)^2} + \frac{\partial O(w,\bar{w})}{z-w} + \cdots$$

$$\bar{T}(\bar{z})\,O(w,\bar{w}) = \cdots + \tilde{h}\frac{O(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} O(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

Def$^{\underline{n}}$: A primary operator O has OPE w/ T & $\bar{T}$ of the form

$$T(z)\,O(w,\bar{w}) = h\frac{O(w,\bar{w})}{(z-w)^2} + \frac{\partial O(w,\bar{w})}{z-w} + \cdots$$

non-sig. terms

$$\bar{T}(\bar{z})\,O(w,\bar{w}) = \tilde{h}\frac{O(w,\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial} O(w,\bar{w})}{\bar{z}-\bar{w}} + \cdots$$

⇒ OPE truncates @ order $(z-w)^{-2}$ & $(\bar{z}-\bar{w})^{-2}$ w/ no higher order poles higher order poles

Since we know the singular parts of OPE of primary operators w\ $T, \bar{T}$, we know how primary operators transform under any conf. transformation.

Transformation rules for primary operators are nice & simple.

Focus on $\delta z = \epsilon(z)$:

$$\delta O(w,\bar{w}) = -\operatorname{Res}\left[\epsilon(z)\, T(z)\, O(w,\bar{w})\right]$$

$$= -\operatorname{Res}\left[\epsilon(z)\left(h\frac{O(w,\bar{w})}{(z-w)^2} + \frac{\partial O(w,\bar{w})}{z-w} + \dots\right)\right]$$

Consider smooth conformal transformations $\ni$ $\epsilon(z)$ has no singularities at $z=w$

$$\Rightarrow \epsilon(z) = \epsilon(w) + (z-w)\,\partial\epsilon(w) + \dots$$

We get the

Transformation rule for primary operators

$$\boxed{\Rightarrow \delta O(w,\bar{w}) = -h\,\partial\epsilon(w)\, O(w,\bar{w}) - \epsilon(w)\,\partial O(w,\bar{w})} \quad (+)$$

& similar for $\delta\bar{z} = \bar{\epsilon}(\bar{z})$

Finite transformation rule comes from integrating up $(+)$:

$$z \to z'(z), \quad \bar{z} \to \bar{z}'(\bar{z})$$

$$O(z,\bar{z}) \to O'(z',\bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h}\left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\tilde{h}} O(z,\bar{z})$$

Spectrum of weights $(h,\tilde{h})$ of primary operators is one of

Spectrum of weights $(h, \tilde{h})$ of primary operators is one of main objects of interest in CFT, similar to mass spectrum of particles in QFT (even related in AdS/CFT)

In statistical mechanics, weights of primary operators $\Updownarrow$ "critical exponents"

(iv) Example: Free scalar field

Let's see some of these concepts in action for free scalar field.

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \, \partial_\alpha X \partial^\alpha X$$

EoM: $\partial^2 X = 0$ classically

Path integral version:

$$0 = \int DX \frac{\delta}{\delta X(\sigma)} e^{-S}$$

total derivative

$$= \int DX \, e^{-S} \left[\frac{1}{2\pi\alpha'} \partial^2 X(\sigma) \right]$$

$$\Rightarrow 0 = \langle \partial^2 X(\sigma) \rangle$$

Ehrenfest's Thm

Propagator:

$$0 = \int DX \frac{\delta}{\delta X(\sigma)} \left[e^{-S} X(\sigma') \right]$$

$$= \int DX \, e^{-S} \left[\frac{1}{2\pi\alpha'} \partial^2 X(\sigma) X(\sigma') + \delta(\sigma - \sigma') \right]$$

$$\Rightarrow \langle \partial^2 X(\sigma) X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma - \sigma')$$

$$\Rightarrow \langle \partial^2 X(\sigma) X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma-\sigma')$$

$$\Rightarrow \partial^2 \langle X(\sigma) X(\sigma') \rangle = -2\pi\alpha' \delta(\sigma-\sigma')$$

Use Green's fct (Stoke's Thm Exercise Sheet 6)

$$\partial^2 \ln(\sigma-\sigma')^2 = 4\pi\, \delta(\sigma-\sigma')$$

$$\Rightarrow \langle X(\sigma) X(\sigma') \rangle = -\frac{\alpha'}{2} \ln(\sigma-\sigma')$$

Singularity as $\sigma \to \sigma'$ is UV divergence like in all field theories.

Singularities as $|\sigma-\sigma'| \to \infty$ is IR divergence & special to $d \leq 2$ dimensions: Waves don't disperse in $d \leq 2$!

We can repeat analysis w/ operator insertions $O_1(\sigma_1) \ldots O_n(\sigma_n)$

If $\sigma_1, \ldots \sigma_n \neq \sigma, \sigma'$, analysis is unaltered

$\Rightarrow$ OPE

$$X(\sigma) X(\sigma') = -\frac{\alpha'}{2} \ln(\sigma-\sigma') + \ldots$$

In $\mathbb{C}$ coordinates, we can use $X(z,\bar{z}) = X(z) + \bar{X}(\bar{z})$

$$\Rightarrow X(z) X(w) = -\frac{\alpha'}{2} \ln(z-w) + \ldots$$

$$\bar{X}(\bar{z}) \bar{X}(\bar{w}) = -\frac{\alpha'}{2} \ln(\bar{z}-\bar{w}) + \ldots$$

NB: $X(z)$ is not a nice operator due to this messy OPE!

However, $\partial X(z)$ has a nice OPE:

$$\partial X(z)\, \partial X(w) = -\frac{\alpha'}{2} \frac{1}{(z-w)^2} + \ldots$$

We want to compute OPE between stress-energy tensor & O_i operators

We want to compute OPE between stress-energy tensor & other operators.

In classical theory,

$$T = -\frac{1}{\alpha'} \partial X \partial X$$

This involves a product $\Rightarrow$ needs care in quantum theory. Let's normal order, as usual, i.e. annihilation operators to the right $\Rightarrow$ vacuum is annihilated

$$\Rightarrow T = -\frac{1}{\alpha'} :\partial X \partial X: = -\frac{1}{\alpha'} \lim_{z \to w} \left(\partial X(z) \partial X(w) - \langle \partial X(z) \partial X(w) \rangle \right)$$

$\Rightarrow \langle T \rangle = 0$ but we did not use creation/annihilation operators explicitly.

We can use Wick's Theorem to compute OPE's:

$$\underbrace{\phi(z) \chi(w)}_{\text{time-ordered}} = \underbrace{\langle \phi(z) \chi(w) \rangle}_{\text{propagator}} + \underbrace{:\phi(z) \chi(w):}_{\text{non-singular!}}$$

For general products of fields, need to consider all possible "contractions"

$$\Rightarrow T(z) \partial X(w) = -\frac{1}{\alpha'} :\partial X(z) \partial X(z): \partial X(w)$$

$$= -\frac{1}{\alpha'} :\partial X(z) \partial X(z) \partial X(w):$$

2 possible contractions $\langle \partial X(z) \partial X(w) \rangle$

$$-\frac{2}{\alpha'} \partial X(z) \langle \partial X(z) \partial X(w) \rangle$$

non-singular

$$= -\frac{1}{\alpha'} :\partial X(z) \partial X(z) \partial X(w):$$

$$= -\frac{1}{\alpha'} :\partial X(z)\,\partial X(z)\,\partial X(w): - \frac{2}{\alpha'}\,\partial X(z)\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}+\dots\right)$$

$$= \frac{\partial X(z)}{(z-w)^2} + \dots$$

weight $h=1$ translation term

$$\Rightarrow T(z)\,\partial X(w) = \frac{\partial X(w)}{(z-w)^2} + \frac{\partial^2 X(w)}{z-w} + \dots$$

$\Rightarrow \partial X$ is primary operator w\ weight $(1,0)$

NB: $\partial^n X$ has weight $(h,\tilde{h}) = (n,0)$ but not primary!

Compute $T(z)\,\partial^2 X(w) = \partial_w\left(T(z)\,\partial X(w)\right)$

$$T(z)\,\partial^n X(w) = \partial_w^{n-1}\left(T(z)\,\partial X(w)\right)$$

$$= \underbrace{\frac{n!\,\partial X(w)}{(z-w)^{n+1}} + \dots}_{\text{not primary!}} + \frac{\overset{h=n}{n}\,\partial^n X(w)}{(z-w)^2} + \frac{\partial^{n+1}X(w)}{z-w} + \dots$$

This result fits classical intuition: each ∂ gives spin $s=1$ & dimension $\Delta=1$ & X does not seem to contribute.

BUT X doesn't have a good not of dimension in CFT

Moreover,

$:e^{ikX}:$ is a primary operator w\ weight $h=\tilde{h}=\frac{\alpha' k^2}{4}$

This is an intrinsically quantum result!

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Let's compute $T(z) :e^{ikX(w)}:$

Consider first

$$\partial X(z) :e^{ikX(w)}: = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \partial X(z) :X(w)^n:$$

$$= \sum_{n=1}^{\infty} \frac{(ik)^n}{(n-1)!} :X(w)^{n-1}: \left(-\frac{\alpha'}{2}\frac{1}{z-w}\right) + \dots$$

$$= -\frac{i\alpha' k}{2}\frac{:e^{ikX(w)}:}{z-w} + \dots$$

$$T(z):e^{ikX(w)}: = -\frac{1}{\alpha'} :\partial X(z)\partial X(z): :e^{ikX(w)}:$$

$$= \frac{\alpha' k^2}{4}\frac{:e^{ikX(w)}:}{(z-w)^2} + \frac{ik :\partial X(z) e^{ikX(w)}:}{z-w} + \dots$$

$$T(z):e^{ikX(w)}: = \frac{\alpha' k^2}{4}\frac{:e^{ikX(w)}:}{(z-w)^2} + \frac{\partial_w :e^{ikX(w)}:}{z-w} + \dots$$

transl. term

$\Rightarrow$ $:e^{ikX(w)}:$ is primary w/ weight $\dfrac{\alpha' k^2}{4}$

From now on, drop $: \, :$

Finally, let's consider TT OPE:

$$T(z)T(w) = \frac{1}{\alpha'^2} :\partial X(z)\partial X(z)::\partial X(w)\partial X(w):$$

$z \to w$

$$= \frac{2}{\alpha'^2}\left(-\frac{\alpha'}{2}\frac{1}{(z-w)^2}\right)^2 - \frac{4}{\alpha'^2}\frac{\alpha'}{2}\frac{:\partial X(z)\partial X(w):}{(z-w)^2} + \dots$$

$$= \frac{}{\alpha'^2}\Big(\frac{}{2\,(z-w)^4} - \frac{}{\alpha'^2\, 2}\,\frac{}{(z-w)^2} + \dots$$

$$= \frac{1/2}{(z-w)^4} + \frac{2\,T(w)}{(z-w)^2} - \frac{2}{\alpha'}\,\frac{\partial^2 X(w)\,\partial X(w)}{z-w} + \dots$$

$$= \underbrace{\frac{1/2}{(z-w)^4}}_{\Downarrow} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots$$

$\Rightarrow T$ is <u>not</u> primary!

But T has a weight $(h, \tilde{h}) = (2, 0)$.