Lecture 9

Monday, 14 December 2020 09:17

## Recap

- Ward identities → transformation laws for operators
- Primary operators: nice transformation properties (most sing. term in OPE is $(z-w)^{-2}$)
- Scaling dimensions & spin of operator in terms of weights $(h, \tilde{h})$

- Computed OPE's using Wick contractions
- $\partial X$ is primary operator for free scalar field $(h,\tilde{h})=(1,0)$
- $:e^{ikX}:$ is primary for free scalar field $h = \frac{\alpha' k^2}{4}$
- $T(z)$ is not primary for free scalar field: includes term $\frac{1/2}{(z-w)^4}$

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## IV – The central charge

We have seen for free scalar field, $T$ is not primary but

$$T(z)T(w) = \frac{1/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

let's understand this & $(z-w)^{-4}$ term for generic CFT.

- $T(z)$ is always an operator of weight $(h,\hat{h})=(2,0)$.
  This follows because energy is given by $H \sim \int d\sigma\, T$.
  $\Rightarrow$ $T$ has scaling dimension 2 & it has spin 2 (sym. traceless tensor)
  $\Rightarrow (h,\hat{h}) = (2,0)$
  Similarly $\bar{T}(\bar{z})$ has $(h,\hat{h}) = (0,2)$.

Similarly $T(z)$ has $(h,\tilde{h}) = (0,2)$.

- Most singular term is $(z-w)^{-4}$.

  Each term in the OPE has dimension $\Delta = 4$.

  $\Rightarrow$ of the form $\dfrac{O_n}{(z-w)^n}$ w\ $\Delta[O_n] = 4-n$

  As we will soon see, unitarity $\Rightarrow$ $h,\tilde{h} \geq 0$ $\forall$ operators

  $\Rightarrow$ Most singular term is $(z-w)^{-4}$ w\ coefficient is an operator w\ $\Delta[O_4] = 0$.

- One can show that coefficient of $(z-w)^{-4}$ term must be constant: $c \in \mathbb{C}$.

  (Follows from looking at transformation properties of $T(z)$)
  See Polchinski Pg 48.

- We cannot have $(z-w)^{-3}$ as this would violate symmetry $T(z)T(w) = T(w)T(z) \Rightarrow z \Leftrightarrow w$ symmetric.

  Taylor series saves the $(z-w)^{-1}$ term:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

$$= \frac{c/2}{(w-z)^4} + \frac{2T(w)}{(w-z)^2} - \frac{\partial T(w)}{w-z} + \ldots$$

$$= \frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\cancel{2}\partial T(z)}{(w-z)^{\cancel{2}}}\cancel{(w-z)} - \frac{\partial T(\cancel{z})}{\cancel{w}-z} + \ldots$$

$$= T(w)T(z)$$

In general

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \ldots$$

$$\tilde{T}(\bar{z})\tilde{T}(\bar{w}) = \frac{\tilde{c}/2}{\ldots} \quad 2\tilde{T}(\bar{w}) \quad \partial\tilde{T}(\bar{w})$$

$$\bar{T}(\bar{z})\,\bar{T}(\bar{w}) = \frac{\tilde{c}/2}{(\bar{z}-\bar{w})^4} + \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\partial\bar{T}(\bar{w})}{\bar{z}-\bar{w}} + \dots$$

$c, \tilde{c}$ are called central charges & we will soon see they are the central charges of the Virasoro algebra.

NB: For single free scalar field, we had $c = \tilde{c} = 1$.
For $D$ free scalar fields, we will have $c = \tilde{c} = D$.
$\Rightarrow$ $c, \tilde{c}$ count the degrees of freedom in a CFT.
$c, \tilde{c}$ are not necessarily integer.

Since we know $TT$ OPE $\Rightarrow$ how $T$ transforms under conf. transformations. In particular, the $(z-w)^{-4}$ term means $T$ is not primary (unless $c=0$).

$$\delta T(w) = -\operatorname{Res}\left[\epsilon(z)\,T(z)\,T(w)\right]$$

$$= -\operatorname{Res}\left[\epsilon(z)\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \dots\right)\right]$$

For non-singular $\epsilon(z)$:

$$\epsilon(z) = \epsilon(w) + \partial\epsilon(w)(z-w) + \frac{1}{2}\partial^2\epsilon(w)(z-w)^2 + \frac{1}{6}\partial^3\epsilon(w)(z-w)^3 + \dots$$

$$\delta T(w) = -\epsilon(w)\,\partial T(w) - 2\partial\epsilon(w)\,T(w) - \frac{c}{12}\partial^3\epsilon(w) \qquad (\dagger)$$

NB: $T(w)$ transforms as if it were primary for infinitesimal conformal transformations s.t. $\partial^3\epsilon = 0$.

The finite version of $(\dagger)$ as $z \to z'(z)$ is:

The finite version of (†)

$$T'(z') = \underbrace{\left(\frac{\partial z'}{\partial z}\right)^{-2}\Big[T(z)}_{\text{behavior of primary}} - \underbrace{\frac{c}{12}\{z', z\}}_{\text{contribution of } \frac{c}{12}\partial^3_\epsilon \text{ term}}\Big]$$

Schwarzian (derivative):

$$\{z', z\} = \left(\frac{\partial^3 z'}{\partial z^3}\right)\left(\frac{\partial z'}{\partial z}\right)^{-1} - \frac{3}{2}\left(\frac{\partial^2 z'}{\partial z^2}\right)^2\left(\frac{\partial z'}{\partial z}\right)^{-2}$$

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Infinitesimal version gives (†).

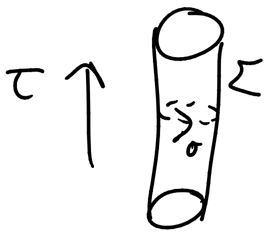
Integrating up (†) whilst preserving group structure of successive conformal transformations gives Schwarzian. Exercise Sheet 7.

We will discuss 3 more properties of the central charge $c$.
For more, see Tong Pg 86, 89-91.

(i) Casimir energy of the string

Let's start tying this back to string theory.

So far we've worked on $\mathbb{C}$ but string worldsheet is cylinder




$\sigma \in [0, 2\pi)$

We can use complex coords: $w = \sigma + i\tau$

$\Sigma$ can be mapped to $\mathbb{C}$ via a conformal transformation:

$$z = e^{-iw}$$

(Exercise Sheet 6, Qu. 1)

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How does $T$ transform?

$$\{z, w\} = 1/2 \quad \text{easily computed}$$

$$T_{cyl}(w) = -z^2 T_{\mathbb{C}}(z) + c/24$$

It makes sense to take $\langle T_{\mathbb{C}} \rangle = 0$ since $\mathbb{C}$ has no spatial scale.

$\Rightarrow \langle T_{cyl} \rangle = c/24 \rightarrow$ gives a Casimir energy!

This is the same result as what we've seen before when we did $\sum_{n>0} n = -\frac{1}{12}$.

$$H = \int d\sigma \, T_{\tau\tau} = -\int d\sigma \left( T_{ww} + \bar{T}_{\bar{w}\bar{w}} \right)$$

$\Rightarrow$ Ground state energy of cylinder

$$E_{cyl} = -\frac{2\pi (c + \tilde{c})}{24}$$

Casimir energy on cylinder

For free scalar field $c = \tilde{c} = 1$, $E/2\pi = -\frac{1}{12}$.

Exactly as we found from summing zero-point energies & removing $\infty$!

## (ii) The Weyl anomaly

Recall, classically, we had $T^{\alpha}{}_{\alpha} = 0$ for CFT.

In quantum theory, we could have an anomaly

$$\langle T^{\alpha}{}_{\alpha} \rangle \neq 0$$

We will now see that

We will now see that

$$\langle T^\alpha{}_\alpha \rangle = -\frac{c}{12} \mathcal{R}_{(2)}, \quad \mathcal{R}_{(2)} \text{ is the Ricci scalar of worldsheet}$$

<u>NB</u>:

- $\langle T^\alpha{}_\alpha \rangle = -\frac{c}{12} \mathcal{R}_{(2)}$ $\forall$ states in CFT.
  Typical of anomalies: come by regulating UV divergences but in UV all finite energy states look the same!
- $\langle T^\alpha{}_\alpha \rangle$ is the same for all states $\Rightarrow$ should depend on background metric (not some operator of the CFT!)
  It should be local & dimension $2 \Rightarrow$ can only be $\mathcal{R}_{(2)}$.

$\Rightarrow \langle T^\alpha{}_\alpha \rangle \sim \mathcal{R}_{(2)}$ but <u>coefficient</u>?

- Using diffeos, can always choose conf. flat metric in 2-d

$$\Rightarrow g_{\alpha\beta} = e^{2\omega} \delta_{\alpha\beta}$$

$$\Rightarrow \mathcal{R}_{(2)} = -2e^{-2\omega} \partial^2 \omega \quad (\#)$$

  So $\langle T^\alpha{}_\alpha \rangle \sim \partial^2 \omega$ & takes different values on different backgrounds related by Weyl transformation.

  $\Rightarrow$ <u>Weyl anomaly</u> or <u>trace anomaly</u>
- Generalisations exist in $d>2$, possibly involving multiple central charges.
- We also have $\langle T^\alpha{}_\alpha \rangle = -\frac{\tilde{c}}{12} \mathcal{R}_{(2)}$.
  So CFT can only be consistently coupled to a fixed curved background metric if $c = \tilde{c}$,

a fixed curved background metric if $c = \tilde{c}$, otherwise <u>gravitational anomaly</u>

- Weyl invariance $\Leftrightarrow$ $c = \tilde{c} = 0$ crucial for string theory as we'll see later.

To compute coefficient in

$$\langle T^\alpha{}_\alpha \rangle = \underline{A}\, R_{(2)}$$

consider varying wrt Weyl transformation.

LHS,

$$\delta_{Weyl} \langle T^\alpha{}_\alpha \rangle = \delta_{Weyl} \int D\phi\, e^{-S}\, T^\alpha{}_\alpha(\sigma)$$

$$= \frac{1}{4\pi} \int D\phi\, e^{-S} \left( T^\alpha{}_\alpha(\sigma) \int d^2\sigma' \sqrt{g}\, \delta_{Weyl}\, g^{\beta\gamma} T_{\beta\gamma}(\sigma') \right)$$

Under Weyl transf. around flat space $\delta g_{\alpha\beta} = 2\omega\, \delta_{\alpha\beta}$

$\Rightarrow \delta g^{\alpha\beta} = -2\omega\, \delta^{\alpha\beta}$

$$\Rightarrow \delta_{Weyl} \langle T^\alpha{}_\alpha(\sigma) \rangle = -\frac{1}{2\pi} \int D\phi\, e^{-S} \left( T^\alpha{}_\alpha(\sigma) \int d^2\sigma'\, T^\beta{}_\beta(\sigma') \right)$$

We need to know $\langle T^\alpha{}_\alpha(\sigma)\, T^\beta{}_\beta(\sigma') \rangle$

In $\mathbb{C}$ coords: $\langle T^\alpha{}_\alpha(\sigma)\, T^\beta{}_\beta(\sigma') \rangle = 16 \underline{\langle T_{z\bar{z}}\, T_{w\bar{w}} \rangle}$

Conservation: $\partial T_{z\bar{z}} = -\bar{\partial} T_{zz}$

$$\Rightarrow \partial_z T_{z\bar{z}}(z,\bar{z})\, \partial_w T_{w\bar{w}}(w,\bar{w}) = \bar{\partial}_{\bar{z}} T_{zz}(z,\bar{z})\, \bar{\partial}_{\bar{w}} T_{ww}(w,\bar{w})$$

$$(*) \qquad = \bar{\partial}_{\bar{z}} \bar{\partial}_{\bar{w}} \left[ \frac{c/2}{(z-w)^4} + \dots \right]$$

term $\sim T\,\partial T$ have spin $\Rightarrow$ vanish in $\langle\ \rangle$!

NB: $\bar{\partial} \frac{1}{(z-w)^4} \neq 0$ due to singularity!

$\underline{\text{NB}}$: $\bar{\partial} \dfrac{1}{(z-w)^4} \neq 0$ due to singularity!

c.f. $\bar{\partial}\partial \ln|z-w|^2 = \partial_{\bar{z}} \dfrac{1}{z-w} = 2\pi\delta(z-w, \bar{z}-\bar{w})$

Propagator

Exercise sheet 6 Qu.2

$$\Rightarrow \partial_{\bar{z}} \partial_{\bar{w}} \frac{1}{(z-w)^4} = \frac{1}{6} \partial_{\bar{z}} \partial_{\bar{w}} \left( \partial_z^2 \partial_w \frac{1}{z-w} \right)$$

$$= \frac{\pi}{3} \partial_z^2 \partial_w \partial_{\bar{w}} \delta(z-w, \bar{z}-\bar{w})$$

Take $\partial_z \partial_w$ out to get ~~(*)~~

$$\langle T_{z\bar{z}}(z,\bar{z}) T_{w\bar{w}}(w,\bar{w}) \rangle = \frac{c\pi}{6} \partial_z \partial_{\bar{w}} \delta(z-w, \bar{z}-\bar{w})$$

$$\Rightarrow \langle T^\alpha{}_\alpha(\sigma) T^\beta{}_\beta(\sigma') \rangle = -\frac{c\pi}{3} \partial^2 \delta(\sigma-\sigma')$$

$$\& \quad \delta_{\text{weyl}} \langle T^\alpha{}_\alpha(\sigma) \rangle = \frac{c}{6} \partial^2 \omega.$$

But, $\delta_{\text{weyl}} R_{(2)} = -2\partial^2\omega$ close to flat space (see H)

$$\Rightarrow \langle T^\alpha{}_\alpha(\sigma) \rangle = -\frac{c}{12} R_{(2)}(\sigma)$$

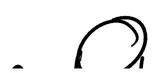
## IV - Commutators & Virasoro Algebra

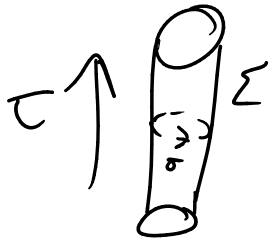
Normally in QFT, we work w/ commutators, e.g. to generate symmetries. So far in CFT we have seen we can determine transformation from OPE.

How are commutators & OPEs related?

### (i) Radial quantisation

Let's first revisit the map from worldsheet $\Sigma$ to $\mathbb{C}$:



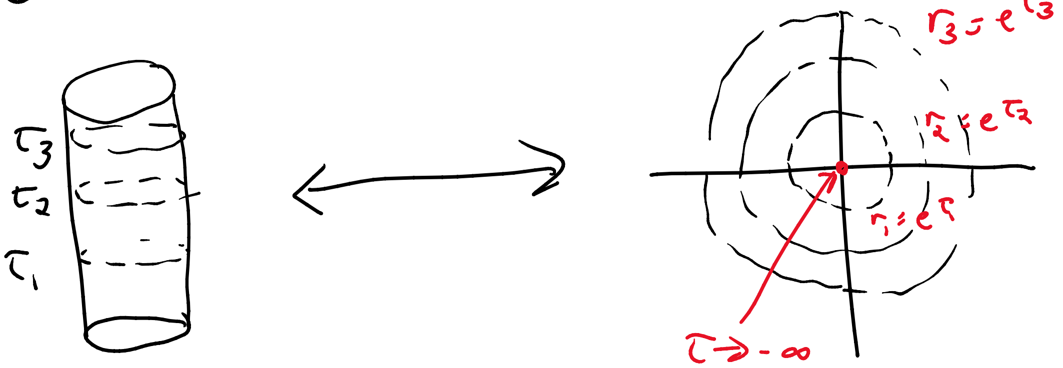



$w = \sigma + i\tau, \ \sigma \in [0, 2\pi)$

Map to $\mathbb{C}$ via $z = e^{-iw}$

Exercise Sheet 6, Qu. 1

Const. $\tau$ slices $\Leftrightarrow$ const. radius circles in $\mathbb{C}$




$\Rightarrow$ Time ordering $\longleftrightarrow$ Radial ordering

Constant time commutation relations $\longleftrightarrow$ Constant radius commutation relations

This is known as radial quantisation.

(ii) Virasoro algebra

$$T_{\text{cylinder}}(w) = -\sum_{n=-\infty}^{\infty} L_n e^{inw} + \frac{c}{24}$$

On the $\mathbb{C}$ plane:

$$T_{\mathbb{C}}(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}}$$ $\leftarrow$ weight $(h, \tilde{h}) = (2, 0)$

$$\overline{T}_{\mathbb{C}}(\overline{z}) = \sum_{n=-\infty}^{\infty} \frac{\tilde{L}_n}{\overline{z}^{n+2}}$$

Inverse from a contour integral:

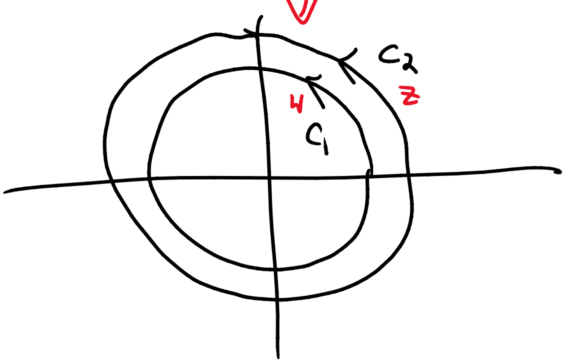
$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z)$$

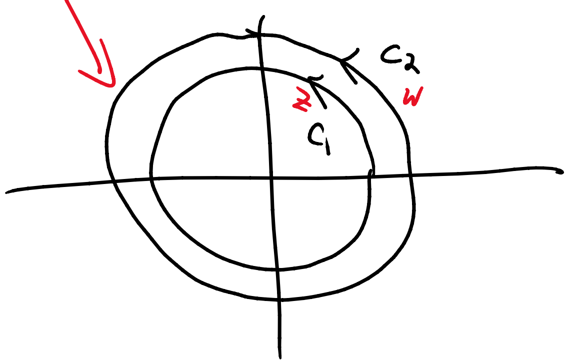
$$\tilde{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\, \bar{z}^{n+1}\, \bar{T}(\bar{z})$$

To compute $[L_m, L_n]$ in CFT, we need to make sure we have the right radial ordering

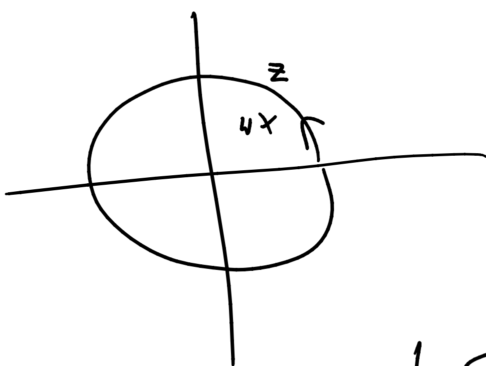
$$[L_m, L_n] = L_m L_n - L_n L_m$$

$$= \left(\oint_{C_1}\frac{dz}{2\pi i}\oint_{C_2}\frac{dw}{2\pi i} - \oint_{C_1}\frac{dw}{2\pi i}\oint_{C_2}\frac{dz}{2\pi i}\right) z^{m+1} w^{n+1} T(z)T(w)$$




Fix w & think about z integral:

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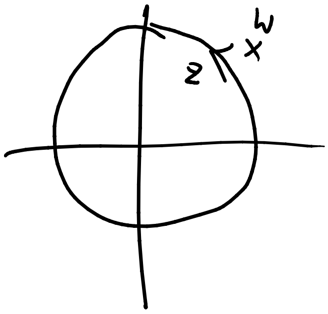


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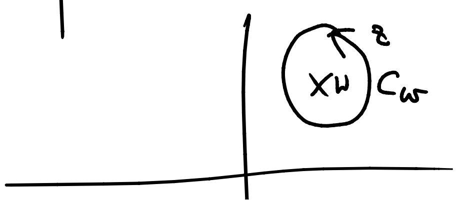
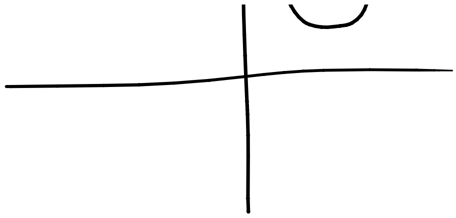


$$=$$ 

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} \oint_{C_w} \frac{dz}{2\pi i} z^{m+1} w^{n+1} T(z)T(w)$$

$$= \oint \frac{dw}{2\pi i} \operatorname*{Res}_{z \to w} \left[ z^{m+1} w^{n+1} \left( \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T(w)}{z-w} \right) \right]$$

Taylor expand around $w$

$$z^{m+1} = w^{m+1} + (m+1) w^m (z-w) + \frac{1}{2} m(m+1) w^{m-1} (z-w)^2 + \frac{1}{6} m(m^2-1) w^{m-2} (z-w)^3 + \ldots$$

$$[L_m, L_n] = \oint \frac{dw}{2\pi i} w^{n+1} \Big[ w^{m+1} \underbrace{\partial T(w)}_{\text{Integrate by parts}} + 2(m+1) w^m T(w) + \frac{c}{12} m(m^2-1) w^{m-2} \Big]$$

$$= (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}$$

Virasoro algebra w\ $c$ = central charge.